On the numerical range of second order elliptic operators with mixed boundary conditions in L^p

Ralph Chill * Hannes Meinlschmidt † Joachim Rehberg ‡

September 8, 2020

We consider second order elliptic operators with real, nonsymmetric coefficient functions which are subject to mixed boundary conditions. The aim of this paper is to provide uniform resolvent estimates for the realizations of these operators on L^p in a most direct way and under minimal regularity assumptions on the domain. This is analogous to the main result in [7]. Ultracontractivity of the associated semigroups is also considered. All results are for two different form domains realizing mixed boundary conditions. We further consider the case of Robin- instead of classical Neumann boundary conditions and also allow for operators inducing dynamic boundary conditions. The results are complemented by an intrinsic characterization of elements of the form domains inducing mixed boundary conditions.

1 Introduction

The regularity of solutions of elliptic or parabolic operators is a classical subject. Uniform estimates for resolvents of elliptic operators and for the semigroups generated by them are central instruments for the study of nonautonomous linear or quasilinear parabolic equations. Much of the theory is standard nowadays and treated in many comprehensive books on parabolic evolution equations; we refer exemplarily to [1, Chapter II], [23, Chapter 6.1], [13] or [29].

*Institut für Analysis, Fakultät Mathematik, TU Dresden, 01062 Dresden, Germany. (ralph.chill@tu-dresden.de)

†Radon Institute for Computational and Applied Mathematics (RICAM), Altenberger Straße 69, 4040 Linz, Austria. (hannes.meinlschmidt@ricam.oeaw.ac.at)

‡Weierstraß Institute for Applied Analysis and Statistics (WIAS), Mohrenstraße 39, 10117 Berlin, Germany (joachim.rehberg@wias-berlin.de)

In this work, we provide uniform resolvent estimates for the L^p-realizations of second order elliptic operators with real, nonsymmetric coefficient functions posed on bounded domains in $\mathbb{R}^d$ and subject to mixed boundary conditions, under minimal regularity assumptions on the domain. In case of smooth domains and real and symmetric coefficient functions, such uniform resolvent estimates are classical ([28, Chapter 7.3]) and have been generalized in [16] to non-smooth domains and mixed boundary conditions. Moreover, the case of non-symmetric coefficient functions has been treated in [7] under pure Dirichlet or pure Neumann- or Robin conditions.

Our main result is a complement to the main result in [7]. We give an (optimal) estimate for the half angle θ_p of the sector containing the numerical range of the L^p-realization of the elliptic operator. The proof given here differs from the proof in [7] and uses ideas from [9]. The estimate for the numerical range immediately yields resolvent estimates outside the sector with half angle θ_p, and these estimates stand in a one-to-one correspondence to the holomorphy of the corresponding semigroup on a sector with half angle $\frac{\pi}{2} - \theta_p$; see [27, Theorem 1.45] for details. This yields an H^∞ functional calculus, $\mathcal{R}$-sectoriality and maximal parabolic regularity for the associated operators. It is known that the obtained resolvent estimates estimates are in general optimal [6,20]. We moreover mention ultracontractivity and associated properties. The results extend to elliptic operators with mixed Robin- and Dirichlet boundary conditions, and are also applied to parabolic evolution equations with dynamical boundary conditions.

All L^p-realizations of elliptic operators are here defined via sesquilinear forms. It turns out that sectoriality of the underlying operator in L^p, the H^∞-functional calculus, $\mathcal{R}$-sectoriality and L^p-maximal regularity solely depend on the properties of the coefficients of the second order elliptic operator and structural properties of the form domain, but neither on the regularity of the domain nor on the type of boundary conditions. On the other hand, ultracontractivity of the underlying semigroup and a characterisation of the trace zero property on (parts of) the boundary for elements from the form domain (see Appendix) solely depend on regularity of the form domain via properties of the domain in $\mathbb{R}^d$ and the type of boundary conditions. They are, however, stable under the passage from mixed Neumann- and Dirichlet boundary conditions to mixed Robin- and Dirichlet boundary conditions, if the part of the boundary where Dirichlet boundary conditions are imposed does not change.

For the sake of readability, we consider only pure second order operators. Moreover, it would also be possible to deal with weighted Lebesgue- and associated Sobolev spaces which would allow for more general and involved differential operators. Since this is rather involved to combine with mixed boundary conditions and already incorporated in [7] in the case of non-mixed boundary conditions, we have decided to not include weighted spaces.

2 Preliminaries

Let $\Omega \subseteq \mathbb{R}^d$ be a domain. We do not require Ω to be bounded and there are no further regularity assumptions on Ω until Section 4. Let us denote the usual (complex) Lebesgue spaces by $L^p(\Omega)$ and the corresponding first order Sobolev spaces, given by all $L^p(\Omega)$ functions whose first-order weak derivatives are again in $L^p(\Omega)$, by $W^{1,p}(\Omega)$.

2.1 Form domain

We first define the considered form domains V. These will be Sobolev spaces incorporating a partially vanishing trace condition leading to associated evolution equations with mixed boundary conditions. Let $D \subseteq \partial\Omega$ be a closed subset of the boundary of Ω, the *D*irichlet boundary part. In all of the following, let V be either of the following spaces:

$$V = W_D^{1,2}(\Omega) := \overline{\underline{W}_D^{1,2}(\Omega)}^{W^{1,2}(\Omega)} \quad \text{or} \quad V = \widetilde{W_D^{1,2}(\Omega)} := \overline{C_D^\infty(\Omega)}^{W^{1,2}(\Omega)}$$

where

$$\underline{W}_D^{1,2}(\Omega) := \{u \in W^{1,2}(\Omega) \colon \operatorname{dist}(\operatorname{supp} u, D) > 0\}$$

and

$$C_D^\infty(\Omega) := \{u \in C^\infty(\Omega) \colon u = v|_\Omega \text{ for } v \in C_c^\infty(\mathbb{R}^d) \text{ with } \operatorname{supp} v \cap D = \emptyset\}.$$

Clearly, for $D = \emptyset$, the former $V = W_D^{1,2}(\Omega)$ is just the usual $W^{1,2}(\Omega)$. In the case $\Omega = \mathbb{R}^d$, the two choices for V coincide by a mollifier argument. Note that $C_c^\infty(\Omega) \subset V$ for both choices of V, so either V is dense in $L^2(\Omega)$. Moreover, both spaces satisfy the following additional properties:

(V_1) V is a sublattice of $W^{1,2}(\Omega)$, that is, for every $u \in V$ one has $(\operatorname{Re} u)^+ \in V$,

(V_2) V is stable under the operation $u \mapsto (|u| \wedge \mathbf{1}) \operatorname{sign} u$.

Indeed, these properties are classical for $V = \widetilde{W_D^{1,2}(\Omega)}$ and for $W^{1,2}(\Omega)$, see [27, Propositions 4.4&4.11]. For $V = W_D^{1,2}(\Omega)$ with $D \neq \emptyset$, they follow from the $W^{1,2}(\Omega)$ case for the dense subspace $\underline{W}_D^{1,2}(\Omega)$ and then by continuity for the whole $W_D^{1,2}(\Omega)$. We mention that in property (V_2), the constant function $\mathbf{1}$ need not be an element of the form domain V.

Remark 2.1. The above definitions of V imply a certain, implicit zero trace property on D for its elements in an abstract way. It is possible to obtain more explicit characterisations of zero traces, at least under certain regularity assumptions. Indeed, in the appendix, we show that under natural, very mild assumptions on D and $\partial\Omega$ and the regularity of the relative boundary of D within $\partial\Omega$ (Assumption 6.1), the space $W^{1,2}_D(\Omega)$ can be characterized by the set of all $u \in W^{1,2}(\Omega)$ which satisfy a Hardy-type inequality w.r.t. D or which satisfy

$$\lim_{r\searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)\cap\Omega} |u| = 0$$

for $\mathcal{H}_{d-1}$-a.e. $x \in D$. Under stronger conditions, $W^{1,2}_D(\Omega)$ and $\widetilde{W^{1,2}_D(\Omega)}$ in fact coincide and can then be characterized intrinsically as mentioned. We refer to e.g. [15, Theorem 2.1]. (We assume such a setup in Section 5 below, see Assumption 5.1.) See moreover also [4, Theorem 8.7 (iii)] in the context of (ε, δ)-domains.

2.2 Coefficient function and form

Let $a = (a_{ij}) \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ be a real, *uniformly elliptic* coefficient function, that is, $\operatorname{Re}\langle a(x)\xi, \xi\rangle \geq \eta\|\xi\|^2$ for every $x \in \Omega$, $\xi \in \mathbb{C}^d$ and some ellipticity constant $\eta > 0$. Here $\langle\cdot,\cdot\rangle$ denotes the usual Hermitian inner product in $\mathbb{C}^d$. It follows from the boundedness and the uniform ellipticity that a is in addition *uniformly sectorial*, that is, there exists an angle $\theta_2 \in [0, \frac{\pi}{2}[$ such that

$$\langle a(x)\xi, \xi\rangle \in \overline{\Sigma_{\theta_2}} \quad (x \in \Omega,\ \xi \in \mathbb{C}^d), \tag{1}$$

where $\Sigma_\theta = \{r\, e^{i\varphi} \colon r \in]0,\infty[\text{ and } \varphi \in]-\theta,\theta[\}$ is the open sector of half-angle θ if $\theta \in]0, \frac{\pi}{2}[$ and $\Sigma_0 = [0,\infty[$ is the positive real axis. Equivalently, the sectoriality means that

$$\bigl|\operatorname{Im}\langle a(x)\xi, \xi\rangle\bigr| \leq \tan\theta_2 \cdot \operatorname{Re}\langle a(x)\xi, \xi\rangle \quad (x \in \Omega,\ \xi \in \mathbb{C}^d). \tag{2}$$

We define the sesquilinear form $\mathfrak{a}\colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}[u,v] = \int_\Omega \langle a\nabla u, \nabla v\rangle \qquad (u, v \in W^{1,2}(\Omega)).$$

Due to the properties (V_1) and (V_2) of V, and by the assumptions on the coefficient function a, the restriction $\mathfrak{a}_V$ of $\mathfrak{a}$ to $V \times V$ is a sub-Markovian form. This means that $\mathfrak{a}_V$ is closed, continuous, and accretive, and the associated operator A_2 on $L^2(\Omega)$ given by

$$\begin{aligned} \operatorname{dom} A_2 &:= \Bigl\{u \in L^2(\Omega) \colon u \in V \text{ and there exists } f \in L^2(\Omega) \text{ such that} \\ &\qquad\qquad \text{for all } v \in V : \mathfrak{a}[u,v] = \int_\Omega f\bar{v}\Bigr\}, \\ A_2 u &:= f, \end{aligned}$$

is the negative generator of a positive, analytic, contraction C_0-semigroup T_2 on $L^2(\Omega)$ which is in addition L^∞-contractive, see [27, Thms. 1.54, 4.2 and 4.9]. The semigroup T_2 extrapolates consistently to a positive contraction semigroup T_p on $L^p(\Omega)$ for every $p \in [1,\infty[$, and the semigroup T_p is analytic if $p > 1$ ([27, Proposition 3.12, p.56/57&96]). Denote by A_p the negative generator of T_p.

Remark 2.2. (a) Both choices for V lead to realizations A_p on $L^p(\Omega)$ of the second order elliptic operator $-\operatorname{div}(a\nabla\cdot)$ equipped with Dirichlet boundary conditions on $D \subseteq \partial\Omega$ and Neumann boundary conditions on $\partial\Omega \setminus D$. In general, $W^{1,2}_D(\Omega)$ induces a stronger form of Neumann conditions on $\partial\Omega \setminus D$ for functions in the domain of A_2. This can be seen for example in the case where Ω is a disc around the origin from which the positive x-axis is removed to form a slit. Then $u \in \operatorname{dom} A_2$ satisfies

$$\partial_{\nu\downarrow} u = \partial_{\nu\uparrow} u = 0 \quad \text{if } V = W^{1,2}_D(\Omega)$$

and

$$[\partial_\nu u] = \partial_{\nu\downarrow} u - \partial_{\nu\uparrow} u = 0 \quad \text{if } V = \widetilde{W^{1,2}_D(\Omega)}$$

along the slit, where the arrows stand for the conormal derivatives w.r.t. a taken from either side and $D = \emptyset$.

(b) There is the nomenclature *good Neumann boundary conditions* for $V = \widetilde{W^{1,2}_D(\Omega)}$ and *Neumann boundary conditions* for $V = W^{1,2}_D(\Omega)$, see [27, Chapter 4], related to the former space being a smaller, i.e., more regular subspace of $W^{1,2}(\Omega)$. For example, the form $\mathfrak{a}_V$ with good Neumann boundary conditions has the advantage of being a regular Dirichlet form, in the sense that $C(\overline{\Omega}) \cap V$ is dense in V.

3 The numerical range

We next determine a sector which includes the numerical range of A_2. First, a preliminary lemma.

Lemma 3.1. *For every $p \in [1,\infty[$, the space* $\operatorname{dom} A_2 \cap \operatorname{dom} A_p \cap L^\infty(\Omega)$ *is a core for A_p, that is, it is dense in* $\operatorname{dom} A_p$ *equipped with the graph norm.*

Proof. The semigroups T_2 and T_p are consistent, that is, $T_2 = T_p$ on $L^2(\Omega) \cap L^p(\Omega)$. By taking Laplace transforms, $(I + A_2)^{-1} = (I + A_p)^{-1}$ on $L^2(\Omega) \cap L^p(\Omega)$. The two resolvents thus also coincide on the smaller space $L^1(\Omega) \cap L^\infty(\Omega)$, which is dense in $L^p(\Omega)$. The resolvent $(I + A_p)^{-1}$ being an isomorphism between $L^p(\Omega)$ and $\operatorname{dom} A_p$ (the latter space being equipped with the graph norm), it maps dense subspaces to dense subspaces. Since $(I + A_p)^{-1}$ maps $L^1(\Omega) \cap L^\infty(\Omega)$ onto a subspace of $\operatorname{dom} A_2 \cap \operatorname{dom} A_p \cap L^\infty(\Omega)$, it follows that the latter space is dense in $\operatorname{dom} A_p$. $\square$

The following result is a central one for this work. It is contained in the proof of Theorem 1.1 in [7], where the authors establish an estimate of the angle of analyticity of the semigroup T_p. (Compare with Corollary 3.7 below.) We give here an alternative proof to the one in [7].

Theorem 3.2. *For every $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and every $p \in [2, \infty[$,*

$$\mathfrak{a}\big[u, |u|^{p-2}u\big] \in \overline{\Sigma_{\theta_p}},$$

where

$$\tan\theta_p = \frac{\sqrt{(p-2)^2 + p^2\tan^2\theta_2}}{2\sqrt{p-1}},$$

and θ_2 is as in (1) or (2).

Proof. The case $p = 2$ follows immediately from the sectoriality assumption on the coefficient function a (see (1)). So we focus on the case $p \in \,]2, \infty[$, here proceeding similarly as in the proof of [9, Lemma 1].

Step 1: Let first $u \in C^\infty(\Omega) \cap W^{1,2}(\Omega) \cap L^\infty(\Omega)$. If $u(x) = 0$ and $h \in \mathbb{R}^d$ is arbitrary, then

$$\langle \nabla(|u|^{p-2}u)(x), h\rangle = \lim_{\lambda \searrow 0} \frac{|u(x+\lambda h) - u(x)|}{\lambda} \big|u(x+\lambda h)\big|^{p-2} = 0,$$

so $\nabla(|u|^{p-2}u)(x) = 0$ if $u(x) = 0$. So define

$$\Omega_0 := \big\{x \in \Omega \colon u(x) \neq 0\big\}.$$

Then

$$\mathfrak{a}\big[u, |u|^{p-2}u\big] = \int_{\Omega_0} \langle a\nabla u, \nabla(|u|^{p-2}u)\rangle.$$

Now set $v := |u|^{\frac{p-2}{2}}u$. Since the supports of v and u agree, v and the functions $|v|^{\frac{2-p}{p}}v = u$ and $|v|^{\frac{p-2}{p}}v = |u|^{p-2}u$ are all continuously differentiable and bounded on Ω_0 with the general derivative formula

$$\nabla(|v|^\alpha v) = \alpha|v|^{\alpha-1}v\nabla|v| + |v|^\alpha\nabla v \quad \text{on } \Omega_0 \quad (\alpha \in \mathbb{R}).$$

Hence

$$\begin{aligned}\mathfrak{a}\big[u, |u|^{p-2}u\big] &= \int_{\Omega_0} \Big\langle a\nabla\big(|v|^{\frac{2-p}{p}}v\big), \nabla\big(|v|^{\frac{p-2}{p}}v\big)\Big\rangle \\ &= \int_{\Omega_0} \langle a\nabla v, \nabla v\rangle - \Big(1 - \tfrac{2}{p}\Big)^2 \int_{\Omega_0} \langle a\nabla|v|, \nabla|v|\rangle\end{aligned}$$

$$+\left(1-\tfrac{2}{p}\right)\left(\int\limits_{\Omega_0}\left\langle a\frac{\overline{v}}{|v|}\nabla v,\nabla|v|\right\rangle-\int\limits_{\Omega_0}\left\langle a\nabla|v|,\frac{\overline{v}}{|v|}\nabla v\right\rangle\right).$$

Here we put (see [27, Proposition 4.4])

$$\phi:=\mathrm{Re}\Big(\frac{\overline{v}}{|v|}\nabla v\Big)=\nabla|v|\quad\text{and}\quad\psi:=\mathrm{Im}\Big(\frac{\overline{v}}{|v|}\nabla v\Big)\quad\text{on }\Omega_0. \tag{3}$$

Then

$$\int\limits_{\Omega_0}\langle a\nabla v,\nabla v\rangle=\int\limits_{\Omega_0}\left\langle a\frac{\overline{v}}{|v|}\nabla v,\frac{\overline{v}}{|v|}\nabla v\right\rangle=\int\limits_{\Omega_0}\langle a(\phi+i\psi),\phi+i\psi\rangle,$$

and therefore

$$\begin{aligned}\mathfrak{a}\big[u,|u|^{p-2}u\big]&=\int\limits_{\Omega_0}\langle a(\phi+i\psi),\phi+i\psi\rangle-\left(1-\tfrac{2}{p}\right)^2\int\limits_{\Omega_0}\langle a\phi,\phi\rangle\\&\quad+\left(1-\tfrac{2}{p}\right)\left(\int\limits_{\Omega_0}\langle a(\phi+i\psi),\phi\rangle-\int\limits_{\Omega_0}\langle a\phi,\phi+i\psi\rangle\right)\\&=\left(1-\left(1-\tfrac{2}{p}\right)^2\right)\int\limits_{\Omega_0}\langle a\phi,\phi\rangle+\int\limits_{\Omega_0}\langle a\psi,\psi\rangle\\&\quad+\frac{2i}{p'}\int\limits_{\Omega_0}\langle a\psi,\phi\rangle-\frac{2i}{p}\int\limits_{\Omega_0}\langle a\phi,\psi\rangle.\end{aligned}$$

Set $\varphi:=\sqrt{\frac{4}{pp'}}\phi$. Decomposing a into its symmetric and antisymmetric part,

$$s:=\frac{a+a^*}{2},\quad\text{and}\quad t:=\frac{a-a^*}{2},$$

and noting that $1-\left(1-\frac{2}{p}\right)^2=\frac{4}{p\,p'}=\frac{4(p-1)}{p^2}$, we obtain

$$\begin{aligned}\mathrm{Re}\,\mathfrak{a}\big[u,u|u|^{p-2}\big]&=\int\limits_{\Omega_0}\big[\langle a\varphi,\varphi\rangle+\langle a\psi,\psi\rangle\big]\\&=\int\limits_{\Omega_0}\Big[\big\|s^{\frac12}\varphi\big\|^2+\big\|s^{\frac12}\psi\big\|^2\Big]\end{aligned} \tag{4}$$

and

$$\begin{aligned}\frac{\sqrt{p-1}}{p}\,\mathrm{Im}\,\mathfrak{a}\big[u,u|u|^{p-2}\big]&=\frac{1}{p'}\int\limits_{\Omega_0}\langle a\psi,\varphi\rangle-\frac{1}{p}\int\limits_{\Omega_0}\langle a\varphi,\psi\rangle\\&=\left(1-\tfrac{2}{p}\right)\int\limits_{\Omega_0}\langle s\psi,\varphi\rangle+\int\limits_{\Omega_0}\langle t\psi,\varphi\rangle.\end{aligned} \tag{5}$$

Hence

$$\begin{aligned}\frac{\sqrt{p-1}}{p}\operatorname{Im}\mathfrak{a}\big[u,u|u|^{p-2}\big] &= \Big(1-\tfrac{2}{p}\Big)\int_{\Omega_0}\Big\langle s^{\frac12}\,\psi, s^{\frac12}\,\varphi\Big\rangle + \int_{\Omega_0}\Big\langle s^{-\frac12}\,t\,s^{-\frac12}\,s^{\frac12}\,\psi, s^{\frac12}\,\varphi\Big\rangle \\ &= \int_{\Omega_0}\Big\langle\Big[(1-\tfrac{2}{p})I + s^{-\frac12}\,t\,s^{-\frac12}\Big]\,s^{\frac12}\,\psi, s^{\frac12}\,\varphi\Big\rangle \\ &\le \frac12\int_{\Omega_0}\Big\|(1-\tfrac{2}{p})I + s^{-\frac12}\,t\,s^{-\frac12}\Big\|\,\Big(\big\|s^{\frac12}\,\psi\big\|^2 + \|s^{\frac12}\,\varphi\|^2\Big).\end{aligned}$$

Since t is skew-symmetric, so is $s^{-\frac12}ts^{-\frac12}$, and one gets

$$\Big\|(1-\tfrac{2}{p})I + s^{-\frac12}\,t\,s^{-\frac12}\Big\| = \sqrt{(1-\tfrac{2}{p})^2 + \|s^{-\frac12}\,t\,s^{-\frac12}\|^2}.$$

Now, by the choice of the angle θ_2 (see especially the estimate (2)), we have for every $\xi\in\mathbb{C}^d$ with $\|s^{\frac12}\xi\| = 1$:

$$\Big|\big\langle s^{-\frac12}ts^{-\frac12}s^{\frac12}\xi, s^{\frac12}\xi\big\rangle\Big| = \big|\langle t\xi,\xi\rangle\big| = \big|\operatorname{Im}\langle a\xi,\xi\rangle\big| \le \tan\theta_2\,\operatorname{Re}\langle a\xi,\xi\rangle = \tan\theta_2.$$

Since it and thus also $is^{-\frac12}ts^{-\frac12}$ are symmetric, taking the supremum over all ξ as before implies $\|s^{-\frac12}\,t\,s^{-\frac12}\| \le \tan\theta_2$, which together with the preceding estimate yields the claim for $u\in C^\infty(\Omega)\cap W^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Step 2: We remove the smoothness assumption. Let $u\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ and pick a smooth sequence $(u_n)\subset C^\infty(\Omega)\cap W^{1,2}(\Omega)$ such that $u_n\to u$ in $W^{1,2}(\Omega)$; this is possible by the Meyers-Serrin theorem ([24, Ch. 1.1.5]). There is a subsequence of (u_n) which converges pointwise almost everywhere to u. We do not relabel subsequences. Since $u\in L^\infty(\Omega)$, we can choose (u_n) to be uniformly bounded with, say, $\|u_n\|_{L^\infty(\Omega)}\le\|u\|_{L^\infty(\Omega)}+1$. Then $(|u_n|^{p-2}u_n)$ is bounded $W^{1,2}(\Omega)$ and admits a weakly convergent subsequence. Since $|u|^{p-2}u$ is the pointwise limit, the weak limit of $(|u_n|^{p-2}u_n)$ is $|u|^{p-2}u$. Hence, with the first step,

$$\overline{\Sigma_{\theta_p}}\ni\mathfrak{a}\big[u_n,|u_n|^{p-2}u_n\big]\longrightarrow\mathfrak{a}\big[u,|u|^{p-2}u\big]$$

and this implies the claim. □

Remark 3.3. (a) The ideas of introducing the function v and of using the splitting in (3) in the proof above are taken from [9], while the use of the operator $s^{-\frac12}ts^{-\frac12}$ is borrowed from [7].

(b) It should not come as a surprise that the evaluation of the expression

$$\frac{1}{p'}\langle a(x)\xi,\chi\rangle - \frac1p\langle a(x)\chi,\xi\rangle = \Big(1-\tfrac{2}{p}\Big)\,\big\langle s(x)\xi,\chi\big\rangle + \big\langle t(x)\xi,\chi\big\rangle$$

for $\xi, \chi \in \mathbb{R}^d$ and $x \in \Omega$ plays a crucial role, cf. (5). It is an artefact of

$$\big\langle a(x)(\chi + i\xi), \tfrac{1}{p'}\chi + \tfrac{i}{p}\xi \big\rangle, \tag{6}$$

namely its imaginary part, thanks to the fact that the coefficient function is supposed to be *real* throughout this work. The expression in (6) has turned out to be very important in the case of complex coefficients; we refer to [5] and [30].

Let A be a closed, linear operator on a Banach space X. The *numerical range* of this operator is the set

$$w(A) := \big\{u^*(Au) \colon u \in \operatorname{dom} A,\ \|u\|_X = 1 \text{ and } u^* \in J(u)\big\},$$

where J is the following, *a priori* set-valued duality map:

$$J(u) := \big\{u^* \in X^* \colon \|u^*\|_{X^*} = 1 \text{ and } u^*(u) = \|u\|_X\big\}.$$

If $X = L^p(\Omega)$ for $p \in \,]1,\infty[$ and $\|u\|_{L^p(\Omega)} = 1$, then $J(u)$ contains only one element u^* which we can identify with $|u|^{p-2}u \in L^{p'}(\Omega)$.

We use Theorem 3.2 to determine the numerical range for the operators A_p associated to the form $\mathfrak{a}_V$.

Theorem 3.4. *Let $p \in [2,\infty[$. Then the numerical range $w(A_p)$ of the operator A_p is contained in the closed sector $\overline{\Sigma_{\theta_p}}$, where*

$$\tan\theta_p = \frac{\sqrt{(p-2)^2 + p^2\tan^2\theta_2}}{2\sqrt{p-1}}$$

with θ_2 as in (1).

Proof. Let $u \in \operatorname{dom} A_p \cap \operatorname{dom} A_2 \cap L^\infty(\Omega)$ with $\|u\|_{L^p(\Omega)} = 1$. We show that in fact $|u|^{p-2}u \in V$. Since $\operatorname{dom} A_2 \subset V$, we have $u \in V \cap L^\infty(\Omega)$.

(a) Let first $V = W^{1,2}_D(\Omega)$. Then there exists a sequence $(u_n) \subset \underline{W}^{1,2}_D(\Omega)$ such that $u_n \to u$ in $W^{1,2}(\Omega)$. Thus, up to a subsequence, $u_n \to u$ pointwise almost everywhere. Due to $u \in V \cap L^\infty(\Omega)$, we can arrange that the approximating sequence is uniformly bounded in $L^\infty(\Omega)$, $\|u_n\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)} + 1$. Since the supports are unchanged, $(|u_n|^{p-2}u_n) \subseteq \underline{W}^{1,2}_D(\Omega) \cap L^\infty(\Omega)$ and the sequence is uniformly bounded in $W^{1,2}(\Omega)$. Thus $|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u$ in $W^{1,2}(\Omega)$ along a subsequence. This implies $|u|^{p-2}u \in V$.

(b) Consider next $V = \widetilde{W^{1,2}_D}(\Omega)$. Again, there exists a sequence $(u_n) \subset C^\infty_D(\Omega)$ such that $u_n \to u$ in $W^{1,2}(\Omega)$. As before, it follows that $(|u_n|^{p-2}u_n) \subset \underline{W}^{1,2}_D(\Omega) \cap L^\infty(\Omega)$ and $|u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u$ in $W^{1,2}(\Omega)$ along a subsequence. It remains to show that in fact $(|u_n|^{p-2}u_n) \subset V$. Let n be fixed. By construction, there is a function $v \in C^\infty_c(\mathbb{R}^d)$ such that $v_{|\Omega} = u_n$ and $\operatorname{supp} v \cap D = \emptyset$. Now choose a mollifier family (ϕ_k) and let $v_k := |v|^{p-2}v * \phi_k$. Then $v_k \in C^\infty_c(\mathbb{R}^d)$ and, for k large enough, $\operatorname{supp} v_k \cap D = \emptyset$. Moreover, $v_{k|\Omega} \to |u_n|^{p-2}u_n$ in $W^{1,2}(\Omega)$. Hence $|u_n|^{p-2}u_n \in V$.

Now, with $|u|^{p-2}u \in V$, we finally have

$$u^*(A_p u) = \int_\Omega (A_p u)|u|^{p-2}\overline{u} = \int_\Omega (A_2 u)|u|^{p-2}\overline{u} = \mathfrak{a}(u, |u|^{p-2}u),$$

so that, by Theorem 3.2, $u^*(A_p u) \in \overline{\Sigma_{\theta_p}}$. The set $\operatorname{dom} A_p \cap \operatorname{dom} A_2 \cap L^\infty(\Omega)$ being a core for A_p by Lemma 3.1, the claim follows from an approximation argument. □

Remark 3.5. Interestingly, the above calculations for the nonsymmetric coefficient function a also reproduce the estimates for the numerical range in case of a *symmetric* coefficient function, see [27, Theorem 3.9]. In this case, $\theta_2 = 0$, and hence $\tan\theta_p = \frac{p-2}{2\sqrt{p-1}}$.

From Theorem 3.4 we immediately deduce several corollaries in a standard way; compare with [28, Ch. 1, Theorem 3.9].

Corollary 3.6. *For every $p \in \,]1,\infty[$ the spectrum of A_p is contained in the closed sector $\overline{\Sigma_{\theta_p}}$ and, for every $z \in \mathbb{C} \setminus \overline{\Sigma_{\theta_p}}$,*

$$\left\|(z - A_p)^{-1}\right\|_{\mathcal{L}(L^p(\Omega))} \le \frac{1}{\operatorname{dist}(z, \Sigma_{\theta_p})} \tag{7}$$

with θ_p as in Theorem 3.4.

Proof. Let $p \in [2,\infty[$. By Theorem 3.4, for every $z \in \mathbb{C} \setminus \overline{\Sigma_{\theta_p}}$ and every $u \in \operatorname{dom} A_p$ with $\|u\|_{L^p(\Omega)} = 1$,

$$\begin{aligned}
\|(z - A_p)u\|_{L^p(\Omega)} &= \|(z - A_p)u\|_{L^p(\Omega)} \, \|u^*\|_{L^p(\Omega)^*} \\
&\ge \left|z\, u^*(u) - u^*(A_p u)\right| \\
&= \left|z - u^*(A_p u)\right| \\
&\ge \operatorname{dist}(z, \Sigma_{\theta_p}) \, \|u\|_{L^p(\Omega)}.
\end{aligned}$$

This inequality shows that $z - A_p$ is injective and has closed range. Since $-1 \in \varrho(A_p)$, a connectedness argument yields $\mathbb{C} \setminus \overline{\Sigma_{\theta_p}} \subseteq \varrho(A_p)$, and then the resolvent estimate follows from the above estimate. The case $p \in \,]1,2[$ follows by duality. □

Corollary 3.7. *For every $p \in \,]1,\infty[$, the semigroup generated by $-A_p$ extends to an analytic contraction semigroup on the sector $\Sigma_{\frac{\pi}{2}-\theta_p}$, where θ_p is as in Theorem 3.4.*

Proof. The claim for $p \geq 2$ follows from Corollary 3.6 (analytic semigroup, [27, Theorem 1.54]) and the Lumer-Phillips theorem (contraction semigroup [28, Ch. 1.4]). The case $p \leq 2$ follows by duality. $\square$

Remark 3.8. It was already observed in [7] that the angle θ_p in the foregoing corollary is optimal. Therefore, also Theorem 3.4 and Corollary 3.6 above are optimal as far as the angle is concerned. An example showing the optimality is provided by the Ornstein-Uhlenbeck semigroup on the weighted space $L^p(\mathbb{R}^d;\mu)$, where μ is the associated invariant Gaussian measure (see [6]).

Corollary 3.9 ([21, Corollary 10.16])**.** *For every $p \in]1,\infty[$, the operator A_p has a bounded H^∞-functional calculus on a sector of angle $< \frac{\pi}{2}$.*

Remark 3.10. Regarding Corollary 3.9, see also [19]. If $\theta_p(\mathrm{H}^\infty)$ denotes the optimal (so, smallest) angle for the H^∞-functional calculus, then, by [21, Corollary 10.12], $\theta_2(\mathrm{H}^\infty) \leq \theta_2$, and by [21, Theorem 12.8], $\theta_p(\mathrm{H}^\infty) = \theta_p(\mathcal{R})$, where the latter is the optimal angle of $\mathcal{R}$-sectoriality. For $\Omega = \mathbb{R}^d$ it follows from [21, Theorem 14.4] that $\theta_p(\mathrm{H}^\infty) \leq \theta_p$, and the previous remark then again shows that this estimate is optimal.

From Corollary 3.9, we also immediately obtain maximal L^q regularity for the operators A_p. We refer to [21, Theorem 1.11], or to [22] for a different approach. Let us emphasize that there is no regularity requirement on Ω.

Corollary 3.11. *For every $1 < p, q < \infty$, the operator A_p has L^q-maximal regularity.*

4 Ultracontractivity and compact resolvents

We next consider ultracontractivity of the semigroups T_p generated by $-A_p$ and associated properties. This requires an assumption on Ω, which is as follows:

Assumption 4.1. The form domain V embeds continuously into $L^\beta(\Omega)$ for some $\beta > 2$.

In fact, Assumption 4.1 is equivalently an assumption on ultracontractivity of the semigroups T_p generated by $-A_p$:

Proposition 4.2 ([2, Theorem 7.3.2])**.** *Assumption 4.1 holds true if and only if the consistent semigroup family T_p generated by $-A_p$ is* ultracontractive, *that is, for all $1 \leq p < q \leq \infty$ there exists a constant $c > 0$ such that*

$$\|T_p(t)\|_{\mathcal{L}(L^p(\Omega)\to L^q(\Omega))} \leq c t^{-\frac{\beta}{\beta-2}(\frac{1}{p}-\frac{1}{q})} \quad (0 < t \leq 1). \tag{8}$$

Note that by a scaling argument we necessarily have $\beta \leq 2^\star := \frac{2d}{d-2}$ in Assumption 4.1, the first-order Sobolev exponent associated to 2. If $d > 2$ and $\beta = 2^\star$, then $\frac{\beta}{\beta-2} = \frac{d}{2}$ in the exponent in (8).

Corollary 4.3. *Suppose that Assumption 4.1 holds true and that $|\Omega| < \infty$. Then the following holds true for $p \in]1, \infty[$:*

(a) The embedding $V \hookrightarrow L^2(\Omega)$ is compact.

(b) The resolvents $(z + A_p)^{-1}$ are compact operators on $L^p(\Omega)$ for every $z \in \varrho(-A_2)$.

(c) The semigroup operators $T_p(t)$ are compact operators on $L^p(\Omega)$ for every $t > 0$.

(d) $\sigma(A_2) = \sigma(A_p)$ and the spectral projections corresponding to the nonzero eigenvalues are independent of p.

Proof. (a) follows from $V \hookrightarrow L^\beta(\Omega)$ as in [11, Lemma 7.1]. Thus, $(\lambda + A_2)^{-1}$ is a compact operator on $L^2(\Omega)$. By compactness propagation via interpolation as in [12, Theorem 1.6.1], $(\lambda + A_p)^{-1}$ is compact for every $\lambda \in \varrho(-A_2)$, which is (b). Ultracontractivity implies that $T_2(t)$ is a Hilbert-Schmidt integral operator and thus compact on $L^2(\Omega)$ for $t > 0$. Thus, (c) can be seen from factoring $T_p(t)$ through $L^2(\Omega)$, see [2, Proposition 7.3.3]. Finally, (d) is [12, Corollary 1.6.2]. □

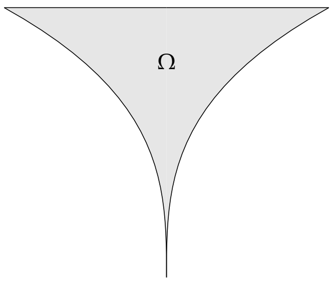

Figure 1: Example of a (non-Sobolev-extension) domain Ω satisfying Assumption 4.1

Remark 4.4. In the case $V = W^{1,2}(\Omega)$, so the largest of the form domains considered in this work, Assumption 4.1 is exhaustively discussed in [24, Section 6.3.4]. See Figure 1 for the exemplary, two dimensional domain $\Omega = \{x \in \mathbb{R}^2 : 0 < x_2 < 1, |x_1| \leq x_2^3\}$ which satisfies Assumption 4.1 for $\beta \leq 4$ ([25, p.1]). As visible there, such a domain Ω may have outward cusps, hence it need be neither a d-set (see (9) below) nor a *homogeneous space* (see [10, Section 2]). Therefore Ω will in general not admit a continuous linear extension operator $E : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^d)$ such that $(Eu)_{|\Omega} = u$ ([17]). There may however be an continuous linear extension operator $V \to W^{1,2}(\mathbb{R}^d)$, see e.g. [14, Sect. 6]. The existence of either extension operator would imply the optimal β in Assumption 4.1. Note moreover that there might exist bounded extension operators $W^{1,2}(\Omega) \to W^{1,r}(\mathbb{R}^d)$ for $r < 2$ or even $W^{1,2}(\Omega) \to W^{\alpha,2}(\mathbb{R}^d)$ for $\alpha < 1$ for domains with outward cusps satisfying Assumption 4.1. Conversely, for r or α sufficiently large, the existence such an extension operator would imply Assumption 4.1. We refer to [33] and the references therein.

5 An extension to Robin and dynamical boundary conditions

We next show how the generality of the foregoing results, in particular Theorem 3.2, can be used to obtain uniform resolvent estimates for differential operators attached to more sophisticated problems. To this end, we need some regularity assumption on Ω and the boundary part D in order to have a well defined trace-type operator. We assume that Ω is bounded throughout this section. The regularity assumption is as follows.

Assumption 5.1. (i) For every point $x \in \Gamma := \overline{\partial\Omega \setminus D}$, there exists an open neighbourhood U_x of x such that $U_x \cap \Omega$ is connected and there exists a continuous linear extension operator $E\colon W^{1,2}(U_x \cap \Omega) \to W^{1,2}(\mathbb{R}^d)$; that is, $(Eu)_{|U_x\cap\Omega} = u$ for every $u \in W^{1,2}(U_x \cap \Omega)$.

(ii) The set Γ is a $(d-1)$-set.

Recall that a Borel set $E \subset \mathbb{R}^d$ is an *N-set* or *N-regular* if there exists a constant $c > 0$ such that

$$cr^N \le \mathcal{H}_N\big(E \cap B_r(x)\big) \le c^{-1}r^N \quad (x \in E,\ r \le 1) \tag{9}$$

where $\mathcal{H}_N$ denotes the N-dimensional Hausdorff measure. We refer to [18, Ch. II.1] for more details.

Remark 5.2. The regularity assumption on $\Gamma = \overline{\partial\Omega \setminus D}$ in Assumption 5.1 is very mild. The required Sobolev extension property is a deeply researched subject. Note that while there D need only be closed, there is a condition on the the relative boundary ∂D of D within $\partial\Omega$, so the transition region between Dirichlet and Neumann boundary parts. Particular cases in which Assumption 5.1 is satisfied include the one where there are Lipschitz charts available around Γ, or, more generally, when Ω is locally an (ε,δ)-domain around Γ. The latter is in fact optimal for $d = 2$. We refer to [14, Section 6.4] for more information.

The immediate consequences of Assumption 5.1 needed in the following are as follows:

(a) There is a bounded linear extension operator which extends $W^{1,2}_D(\Omega)$ to $W^{1,2}_D(\mathbb{R}^d)$. This can be seen by inspecting the the proofs of [14, Theorem 6.9] and [14, Proposition 6.5] and observing that they stay correct for $W^{1,2}_D(\Omega)$ by changing $C^\infty_D(\Omega)$ to $\underline{W}^{1,2}_D(\Omega)$. In particular,

$$W^{1,2}_D(\Omega) \subseteq \widetilde{W^{1,2}_D}(\Omega)$$

and the spaces in fact coincide. Indeed, extend $u \in W^{1,2}_D(\Omega)$ to $U \in W^{1,2}_D(\mathbb{R}^d)$ and approximate U by $(U_n) \subseteq C^\infty_D(\mathbb{R}^d)$ in the $W^{1,2}(\mathbb{R}^d)$ norm. Then the restrictions $u_n := (U_n)_{|\Omega}$ are elements of $C^\infty_D(\Omega)$ and approximate u in the $W^{1,2}(\Omega)$ norm.

(b) There is a well defined trace map $\operatorname{tr}\colon V \to L^\beta(\Gamma; \mathcal{H}_{d-1})$, where $\beta > 2$ ([3]).

Hence, for nonnegative $b \in L^\infty(\Gamma;\mathcal{H}_{d-1})$, the form $\mathfrak{b}\colon V \times V \to \mathbb{C}$ given by

$$\mathfrak{b}(u,v) := \mathfrak{a}(u,v) + \int_\Gamma b\,(\operatorname{tr} u)(\overline{\operatorname{tr} v})\; d\mathcal{H}_{d-1} \quad (u,v \in V),$$

is well defined, continuous, closed and accretive. In fact, it is even a sub-Markovian form. The operator B_2 on $L^2(\Omega)$ associated with this form is the negative generator of an analytic contraction semigroup S_2 which extends consistently to contraction semigroups S_p on all $L^p(\Omega)$-spaces, $p \in [1,\infty[$. The negative generator of S_p is denoted by B_p. All these properties follow as in Section 2.

The operators B_p are realizations of the elliptic operator $-\operatorname{div}(a\,\nabla\cdot)$ with mixed Dirichlet and Robin boundary conditions. The corresponding parabolic evolution problem associated with B_p is formally

$$\begin{aligned} u_t - \operatorname{div}(a\,\nabla u) &= f && \text{in } (0,\infty)\times\Omega,\\ u &= 0 && \text{on } (0,\infty)\times D,\\ \langle a\nabla u, \nu\rangle + bu &= 0 && \text{on } (0,\infty)\times(\partial\Omega\setminus D),\\ u(0,\cdot) &= u_0 && \text{in } \Omega, \end{aligned}$$

where ν is the unit outer normal. That is, one has Dirichlet boundary conditions on D and Robin boundary conditions on $\partial\Omega\setminus D$, which reduce to Neumann boundary conditions on the set $[b=0]$.

Since

$$\int_\Gamma b\,(\operatorname{tr} u)(\operatorname{tr}|u|^{p-2}\bar{u})\; d\mathcal{H}_{d-1} = \int_\Gamma b\operatorname{tr}(|u|^p)\; d\mathcal{H}_{d-1} \geq 0$$

for every $u \in V \cap L^\infty(\Omega)$, by Theorem 3.2, the numerical range of the operator B_p is contained in the same sector as the numerical range of the operator A_p. From Theorem 3.4, Corollary 3.6 and the proof of Corollary 3.7, we thus obtain the following result. (Ultracontractivity is inferred from the $W^{1,2}$-extension property of V, see Remark 4.4, and Proposition 4.2.)

Theorem 5.3. *For every $p \in [2,\infty[$, the numerical range of B_p is contained in the sector $\overline{\Sigma_{\theta_p}}$, where θ_p is as in Theorem 3.4. Moreover, for every $p \in\,]1,\infty[$,*

$$\left\|(z-B_p)^{-1}\right\|_{\mathcal{L}(L^p(\Omega))} \leq \frac{1}{\operatorname{dist}(z,\Sigma_{\theta_p})} \tag{10}$$

for every $z \in \mathbb{C}\setminus\overline{\Sigma_{\theta_p}}$ and the semigroup generated by $-B_p$ extends to an analytic contraction semigroup on the sector $\Sigma_{\frac{\pi}{2}-\theta_p}$ and is ultracontractive.

It is also possible to treat dynamical boundary conditions in this approach. Fix a measurable subset $S \subseteq \Gamma$. Then the embedding $j\colon V \to L^2(\Omega)\times L^2(S;\mathcal{H}_{d-1})$ defined by $u \mapsto (u,\operatorname{tr} u)$ is continuous, injective and has dense range, see [31, Lemma 2.10]. Via this embedding, the form $(\mathfrak{b},V)$ induces also an operator $\hat{B}_2$ on the Hilbert space

$H = L^2(\Omega) \times L^2(S;\mathcal{H}_{d-1})$. By [31, Proposition 2.16], the form $\mathfrak{b}$ is sub-Markovian, so that $-\hat{B}_2$ generates an analytic contraction semigroup $\hat{S}_2$ which extends consistently to contraction semigroups $\hat{S}_p$ on all $L^p(\Omega) \times L^p(S;\mathcal{H}_{d-1})$-spaces, $p \in [1,\infty[$. The negative generator of $\hat{S}_p$ is denoted by $\hat{B}_p$.

The corresponding parabolic evolution problem associated with $\hat{B}_p$ is formally

$$\begin{aligned} u_t - \operatorname{div}(a\,\nabla u) &= f && \text{in } (0,\infty)\times\Omega,\\ u &= 0 && \text{on } (0,\infty)\times D,\\ u_t + \langle a\nabla u,\nu\rangle + bu &= g && \text{on } (0,\infty)\times S,\\ \langle a\nabla u,\nu\rangle + bu &= 0 && \text{on } (0,\infty)\times(\partial\Omega\setminus(D\cup S)),\\ u(0,\cdot) &= u_0 && \text{in } \Omega, \end{aligned}$$

that is, one has Dirichlet boundary conditions on D, dynamical boundary conditions on S, and Robin boundary conditions on $\partial\Omega \setminus (D\cup S)$, which reduce to Neumann boundary conditions on the set $[b = 0]$. Since $\hat{B}_p$ is again fundamentally linked to the form $\mathfrak{a}$, the result about the numerical range transfers immediately from Theorem 3.2. Regarding ultracontractivity, we refer to continuity of the trace operator $\operatorname{tr}\colon V \to L^\beta(S;\mathcal{H}_{d-1})$ where $\beta > 2$ and the reasoning in [31, Lemma 2.19].

Theorem 5.4. *For every $p \in [2,\infty[$, the numerical range of $\hat{B}_p$ is contained in the sector $\overline{\Sigma_{\theta_p}}$, where θ_p is as in Theorem 3.4. Moreover, for every $p \in]1,\infty[$,*

$$\left\|(z-\hat{B}_p)^{-1}\right\|_{\mathcal{L}(L^p(\Omega))} \le \frac{1}{\operatorname{dist}(z,\Sigma_{\theta_p})} \tag{11}$$

for every $z \in \mathbb{C}\setminus\overline{\Sigma_{\theta_p}}$ and the semigroup generated by $-\hat{B}_p$ extends to an analytic contraction semigroup on the sector $\Sigma_{\frac{\pi}{2}-\theta_p}$ and is ultracontractive.

Remark 5.5. It would also be possible to include a $(d-1)$-regular hyperplane $\Sigma \subset \Omega$ in S in a straightforward manner. This would then lead to a dynamic “jump condition”

$$u_t + \left[\langle \mathfrak{a}\nabla u, \nu_\Sigma\rangle\right] = h \quad \text{on } (0,\infty)\times\Sigma.$$

We refer to [31].

6 Appendix: Intrinsic characterisation for the form domain

In this section, we give a completely *intrinsic* characterisation for $V = W^{1,2}_D(\Omega)$, corresponding to the philosophy in [32], see especially Remark 4 there. In fact, we do so for the full scale $W^{1,p}_D(\Omega)$ with $p \in]1,\infty[$. We suppose that Ω is bounded and let $p \in]1,\infty[$ be fixed throughout this section. The characterisation is given under following very mild assumption on $\partial\Omega$ and D which we assume to hold for the rest of this appendix:

Assumption 6.1. (i) For every point $x \in \partial D$, the relative boundary of D within $\partial\Omega$, there exists an open neighbourhood U_x of x such that $U_x \cap \Omega$ is connected and there exists a continuous linear extension operator $E\colon W^{1,p}(U_x \cap \Omega) \to W^{1,p}(\mathbb{R}^d)$; that is, $(Eu)_{|U_x\cap\Omega} = u$ for all $u \in W^{1,p}(U_x \cap \Omega)$.

(ii) The boundary $\partial\Omega$ and the set D itself are $(d-1)$-sets.

Remark 6.2. Comparing to Assumption 5.1—which we do *not* suppose to hold for this section—, the Sobolev extension condition is required *only* on the relative boundary ∂D of D within $\partial\Omega$. Thus, the remaining part of $\partial\Omega \setminus \partial D$ might be highly irregular in a topological sense. We do however suppose the measure-theoretic condition that $\partial\Omega$ and D are $(d-1)$-regular in Assumption 6.1, which is not included in Assumption 5.1 and which effectively means that $\Gamma = \partial\Omega \setminus D$ is also $(d-1)$-regular.

For closed $E \subseteq \partial\Omega$, we define the spaces

$$\underline{W}^{1,p}_E(\Omega) := \{u \in W^{1,p}(\Omega)\colon \operatorname{dist}(\operatorname{supp} u, E) > 0\},$$

and

$$C^\infty_E(\Omega) := \{u \in C^\infty(\Omega)\colon u = v|_\Omega \text{ for } v \in C^\infty_c(\mathbb{R}^d),\ \operatorname{supp} v \cap E = \emptyset\},$$

and their closures in $W^{1,p}(\Omega)$:

$$W^{1,p}_E(\Omega) := \overline{\underline{W}^{1,p}_E(\Omega)}^{W^{1,p}(\Omega)} \quad\text{and}\quad \widetilde{W^{1,p}_E}(\Omega) := \overline{C^\infty_E(\Omega)}^{W^{1,p}(\Omega)}.$$

We have already seen the latter two spaces in the previous sections in the special case $p = 2$. The characterisation of $W^{1,p}_D(\Omega)$ is as follows. (We use $\operatorname{dist}_D(x) := \operatorname{dist}(x, D)$.)

Theorem 6.3. *Let $u \in W^{1,p}(\Omega)$. The following are equivalent.*

(i) $u \in W^{1,p}_D(\Omega)$.

(ii) $u/\operatorname{dist}_D \in L^p(\Omega)$.

(iii) For $\mathcal{H}_{d-1}$-almost every $x \in D$,

$$\lim_{r\searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)\cap\Omega} |u| = 0.$$

Remark 6.4. If one and thus all of the conditions in Theorem 6.3 hold true, then we have a *Hardy inequality* for elements of $W^{1,p}_D(\Omega)$:

$$\left(\int_\Omega \left|\frac{u}{\operatorname{dist}_D}\right|^p\right)^{\frac{1}{p}} \lesssim \|u\|_{W^{1,p}(\Omega)} \quad (u \in W^{1,p}_D(\Omega)).$$

In particular,

$$u \mapsto \|u\|_{W^{1,p}(\Omega)} + \left\|\frac{u}{\operatorname{dist}_D}\right\|_{L^p(\Omega)}$$

is an equivalent norm on $W^{1,p}_D(\Omega)$.

A consequence of the characterisation of $W^{1,p}_D(\Omega)$ in Theorem 6.3 is that the constant one function $\mathbf{1}$ is not an element of that space if $D \neq \emptyset$. The proof follows after the one of Theorem 6.3 below.

Corollary 6.5. *Let $\mathbf{1} \in W^{1,p}(\Omega)$ denote the constant one function. If $D \neq \emptyset$, then $\mathbf{1} \notin W^{1,p}_D(\Omega)$.*

We next state a preliminary geometric lemma which will allow us to prove Theorem 6.3 by reducing it to a similar characterisation theorem in a more regular situation, Proposition 6.8 below. It says that a subset of a regular set can be extended to a regular set in an arbitrarily small manner. We state and prove it for a general bounded N-regular set Λ where $0 < N \leq d$. The proof relies on a sort of dyadic decomposition for regular sets established by David and refined by Christ and is given at the very end of the paper.

Lemma 6.6. *Let $\Lambda \subset \mathbb{R}^d$ be bounded and N-regular. Let further $\Xi \subseteq \Lambda$ and $\rho > 0$. Then there exists an N-set $\Xi^\bullet$ such that $\Xi \subseteq \Xi^\bullet \subseteq \Lambda$ and $\sup\{\operatorname{dist}(x, \Xi) \colon x \in \Xi^\bullet \setminus \Xi\} \leq \rho$.*

Corollary 6.7. *There exists a closed $(d-1)$-set $\Upsilon \subseteq \partial\Omega$ such that $\operatorname{dist}(\Upsilon, D) > 0$ and for every point $x \in \overline{\partial\Omega \setminus (D \cup \Upsilon)}$ there is an open neighbourhood U_x of x such that $U_x \cap \Omega$ is connected and there exists a continuous linear extension operator $E \colon W^{1,p}(U_x \cap \Omega) \to W^{1,p}(\mathbb{R}^d)$.*

Proof. By Assumption 6.1, for every $x \in \partial D$ there exists an open $W^{1,p}$-extension neighbourhood U_x of x. The family $(U_x)_{x \in \partial D}$ is then an open covering of ∂D. By compactness, it thus admits a finite subcovering $(U_{x_j})_j$.

Now choose $\varepsilon > 0$ such that $\{x \in \mathbb{R}^d \colon \operatorname{dist}(x, \partial D) < 3\varepsilon\} \subseteq \bigcup_j U_{x_j}$, and set $C := \{x \in \partial\Omega \setminus D \colon \operatorname{dist}(x, \partial D) \geq 2\varepsilon\}$. Clearly, for every $x \in \partial\Omega \setminus \overline{(D \cup C)}$ there is an open $W^{1,p}$-extension neighbourhood. Let $C^\bullet$ be a $(d-1)$ regular set containing C with $\sup\{\operatorname{dist}(x, C) \colon x \in C^\bullet \setminus C\} \leq \varepsilon$ and define $\Upsilon := \overline{C^\bullet}$; such a set exists by Lemma 6.6. Then Υ has the required properties; for the $(d-1)$ property, see [18, Ch. VIII Proposition 1]. $\square$

With the foregoing result, we can now make use of the characterisation of a zero trace property for a more regular situation in [15] which we quote adapted to our setting:

Proposition 6.8 ([15, Theorem 2.1])**.** *Let $\Upsilon \subseteq \partial\Omega$ be a closed $(d-1)$-set such that for every point $x \in \overline{\partial\Omega \setminus (D \cup \Upsilon)}$ there is an open neighbourhood U_x of x such that $U_x \cap \Omega$ is connected and there exists a continuous linear extension operator $E \colon W^{1,p}(U_x \cap \Omega) \to W^{1,p}(\mathbb{R}^d)$. Let $u \in W^{1,p}(\Omega)$. Then the following are equivalent:*

(i) $u \in \widetilde{W^{1,p}_{D \cup \Upsilon}}(\Omega)$.

(ii) $u / \operatorname{dist}_{D \cup \Upsilon} \in L^p(\Omega)$.

(iii) For $\mathcal{H}_{d-1}$-a.e. $x \in D \cup \Upsilon$,

$$\lim_{r \searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)\cap\Omega} |u| = 0.$$

We prove Theorem 6.3 and Corollary 6.5.

Theorem 6.3. Choose $\Upsilon \subseteq \partial\Omega$ as in Corollary 6.7 and a cut-off function $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\eta \cap \Upsilon = \emptyset$ and $\eta = 1$ in a neighbourhood of D. Write $u = (1-\eta)u + \eta u$. Clearly, $(1-\eta)u \in \underline{W}_D^{1,p}(\Omega)$, so $(1-\eta)u/\operatorname{dist}_D \in L^p(\Omega)$ and

$$\lim_{r \searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)\cap\Omega} |(1-\eta)u| = 0$$

for all $x \in D$. It is thus sufficient to prove all equivalences for ηu only.

((i) $\implies$ (ii)): Assume $\eta u \in W_D^{1,p}(\Omega)$. Choose a sequence $(u_k) \subset \underline{W}_D^{1,p}(\Omega)$ approximating u in $W^{1,p}(\Omega)$. We have $\eta u_k/\operatorname{dist}_{D\cup\Upsilon} \in L^p(\Omega)$ and $\eta u_k \to \eta u$ in $W^{1,p}(\Omega)$. Proposition 6.8 implies that the set of $v \in W^{1,p}(\Omega)$ satisfying $v/\operatorname{dist}_{D\cup\Upsilon} \in L^p(\Omega)$ is *closed* in $W^{1,p}(\Omega)$. Hence $\eta u/\operatorname{dist}_D \in L^p(\Omega)$.

((ii) $\implies$ (i)): Assume $\eta u/\operatorname{dist}_D \in L^p(\Omega)$. Then $\eta u/\operatorname{dist}_{D\cup\Upsilon} \in L^p(\Omega)$ and $\eta u \in W_{D\cup\Upsilon}^{1,p}(\Omega)$ by Proposition 6.8. In particular, ηu in $W^{1,p}(\Omega)$ can be approximated by a sequence of functions from $C_{D\cup\Upsilon}^\infty(\Omega)$. But $C_{D\cup\Upsilon}^\infty(\Omega) \subset \underline{W}_D^{1,p}(\Omega)$, so $\eta u \in W_D^{1,p}(\Omega)$.

((ii) $\iff$ (iii)): Apply Proposition 6.8 to ηu. □

Corollary 6.5. Suppose that $\mathbf{1} \in W_D^{1,p}(\Omega)$. Then, by Theorem 6.3,

$$\lim_{r \searrow 0} \frac{1}{|B(y,r)|} \int_{B(r,y)\cap\Omega} \mathbf{1} = \lim_{r \searrow 0} \frac{|B(y,r)\cap\Omega|}{|B(y,r)|} = 0 \tag{12}$$

for $\mathcal{H}_{d-1}$-almost every $y \in D$. We will show that this leads to a contradiction.

Let $x \in \partial D$, the relative boundary of D within $\partial\Omega$. By Assumption 6.1, there exists an open neighbourhood U_x of x such that $U_x \cap \Omega$ has the $W^{1,p}$-extension property. A domain with the $W^{1,p}$-extension property is necessarily d-regular by a fundamental result by Hajłasz, Koskela and Tuominen [17], so there is a constant $c > 0$ such that

$$|B(y,r)\cap\Omega\cap U_x| \ge cr^d \quad (y \in \Omega \cap U_x,\ r \le 1).$$

This property also holds for $y \in \partial\Omega \cap U_x \subset \partial(\Omega\cap U_x)$. Indeed, let $r \le 1$ and choose $z \in B(y, r/2) \cap \Omega \cap U_x$. Then $B(z, r/2)\cap\Omega\cap U_x \subset B(y,r)\cap\Omega\cap U_x$, hence

$$|B(y,r)\cap\Omega\cap U_x| \ge c2^{-d}r^d \quad (y \in \partial\Omega\cap U_x,\ r \le 1). \tag{13}$$

Now let $0 < r_0 \leq 1$ be such that $B(x, 2r_0) \subset U_x$. Consider $y \in B(x, r_0) \cap D$. Then $B(y, r_0) \subset U_x$, so there is a constant $c_0 > 0$ such that

$$\frac{|B(y,r) \cap \Omega|}{|B(y,r)|} = \frac{|B(y,r) \cap \Omega \cap U_x|}{|B(y,r)|} \geq c_0 > 0$$

for all $r \leq r_0$ by (13). By (12), this is only possible if $\mathcal{H}_{d-1}(B(x,r_0) \cap D) = 0$. But D is $(d-1)$-regular, so $\mathcal{H}_{d-1}(B(x,r_0) \cap D) > 0$. This is the contradiction. □

Proof of Lemma 6.6

In this final subsection, we prove Lemma 6.6. As already mentioned above, we consider N-regular sets where $0 < N \leq d$, and the proof relies on the following Christ decomposition for regular sets:

Theorem 6.9 ([8, Theorem 11]). *Let $\Lambda \subset \mathbb{R}^d$ be bounded and N-regular. Then there exists a collection of relatively open sets $\{Q^k_\alpha \subseteq \Lambda \colon k \in \mathbb{N}_0,\ \alpha \in I_k\}$, where I_k is an index set for every $k \in \mathbb{N}_0$, and constants $\delta \in]0,1[$, $a_0 > 0$, $c_1 < \infty$ such that the following hold true:*

(i) $\mathcal{H}_N\big(\Lambda \setminus \bigcup_{\alpha \in I_k} Q^k_\alpha\big) = 0$ *for every* $k \in \mathbb{N}_0$,

(ii) if $\ell, k \in \mathbb{N}_0$ *and* $\ell \geq k$, *then for every* $\alpha \in I_k$ *and* $\beta \in I_\ell$, *either* $Q^\ell_\beta \cap Q^k_\alpha = \emptyset$ *or* $Q^\ell_\beta \subseteq Q^k_\alpha$,

(iii) if $k \in \mathbb{N}_0$ *and* $\alpha, \beta \in I_k$ *with* $\alpha \neq \beta$, *then* $Q^k_\beta \cap Q^k_\alpha = \emptyset$,

(iv) for every $k \in \mathbb{N}_0$ *and* $\alpha \in I_k$, *there holds* $\operatorname{diam}(Q^k_\alpha) \leq c_1 \delta^k$,

(v) for every $k \in \mathbb{N}_0$ *and* $\alpha \in I_k$, *there is* $z^k_\alpha \in \Lambda$ *such that* $B(z^k_\alpha, a_0\delta^k) \cap \Lambda \subseteq Q^k_\alpha$.

In fact, the Christ decomposition as established in [8, Theorem 11] has some more properties and is valid for *locally doubling metric measure spaces* in general; the "locally" part is due to Morris [26, Proposition 4.2]. We have just extracted the necessary bits needed to prove Lemma 6.6 which we repeat:

Lemma 6.6 *Let $\Lambda \subset \mathbb{R}^d$ be bounded and N-regular. Let further $\Xi \subseteq \Lambda$ and $\rho > 0$. Then there exists an N-set $\Xi^\bullet$ such that $\Xi \subseteq \Xi^\bullet \subseteq \Lambda$ and* $\sup\{\operatorname{dist}(x, \Xi) \colon x \in \Xi^\bullet \setminus \Xi\} \leq \rho$.

Proof. Consider the Christ decomposition of Λ and its data as stated in Theorem 6.9. Let $M \in \mathbb{N}_0$ be so large that $2c_1\delta^M \leq \rho \wedge 1$ and define

$$\Xi^\bullet := \Xi \cup \big\{Q^M_\alpha \colon \alpha \in I_M,\ Q^M_\alpha \cap \Xi_{c_1\delta^M} \neq \emptyset\big\},$$

where

$$\Xi_{c_1\delta^M} := \big\{z \in \Lambda \colon \operatorname{dist}(z, \Xi) \leq c_1\delta^M\big\}.$$

By the choice of M and property (iv) of the Christ decomposition, we already have $\sup\{\operatorname{dist}(x,\Xi) \colon x \in \Xi^\bullet \setminus \Xi\} \le \rho$. We show that $\Xi^\bullet$ is N-regular. Since $\Xi^\bullet \subseteq \Lambda$ and the latter is N-regular, the upper estimate $\mathcal{H}_N(\Xi^\bullet \cap B(x,r)) \lesssim r^N$ for all $x \in \Xi^\bullet$ and $r \le 1$ as in (9) is for free.

For the lower estimate, we will repeatedly make use of the following useful consequences of properties of the Christ decomposition in Theorem 6.9. For convenience, let $\mathcal{Q}_\infty := \bigcap_{k\in\mathbb{N}_0} \bigcup_{\alpha \in I_k} Q_\alpha^k$.

(a) For every $y \in \Lambda$ and every $0 < r \le 1$, there exists $z \in B(y,r) \cap \mathcal{Q}_\infty$.

Proof By property (i) of the Christ decomposition, $\mathcal{H}_N(\Lambda \setminus \mathcal{Q}_\infty) = 0$. Thus, from N-regularity of Λ it follows that $\Lambda \cap B(y,r)$ cannot be a subset of $\Lambda \setminus \mathcal{Q}_\infty$, so $B(y,r) \cap \mathcal{Q}_\infty \neq \emptyset$, for any $0 < r \le 1$.

(b) For every $x \in \Xi^\bullet$ and every $r \le c_1\delta^M$, there exists $z \in B(x,r) \cap \mathcal{Q}_\infty$ and for the unique $\alpha \in I_M$ such that $z \in Q_\alpha^M$ we have $Q_\alpha^M \subseteq \Xi^\bullet$.

Proof Suppose first that $x \in \Xi$ and let, by (a), $z \in B(x,r) \cap \mathcal{Q}_\infty$. Then there exists $\alpha \in I_M$ such that $z \in Q_\alpha^M$. Moreover, $z \in \Xi_{c_1\delta^M}$ due to $r \le c_1\delta^M$, so $Q_\alpha^M \subseteq \Xi^\bullet$ by definition of $\Xi^\bullet$. Now suppose $x \in \Xi^\bullet \setminus \Xi$. By construction, $\Xi^\bullet$ is a union of Ξ and dyadic "cubes" of the M-th generation, so there must be $\alpha \in I_M$ such that $x \in Q_\alpha^M \subseteq \Xi^\bullet$. Since Q_α^M is relatively open, there is r_x such that $\Lambda \cap B(x,r_x) \subseteq Q_\alpha^M$. Now choose $z \in B(x,\min(r,r_x)) \cap \mathcal{Q}_\infty$ in (a), which is then an element of $Q_\alpha^M \subseteq \Xi^\bullet$. Uniqueness of α is a consequence of property (iii) of the Christ decomposition in both cases.

(c) As a consequence of property (v) of the Christ decomposition and N-regularity of Λ, there is a constant $c > 0$ such that

$$\mathcal{H}_N(Q_\alpha^k) \ge \mathcal{H}_N\big(B(z_\alpha^k, a_0\delta^k) \cap \Lambda\big) \ge c(a_0\delta^k)^N \quad (\alpha \in I_k) \tag{14}$$

for all $k \in \mathbb{N}_0$ such that $a_0\delta^k \le 1$.

Now for the actual proof. Let $x \in \Xi^\bullet$.

First suppose that $2c_1\delta^M < r \le 1$. Choose $z \in B(x, c_1\delta^M) \cap \mathcal{Q}_\infty$ with $\alpha \in I_M$ such that $z \in Q_\alpha^M \subseteq \Xi^\bullet$ as in (b). Then $B(z,r/2) \subset B(x,r)$ due to $|x-z| < c_1\delta^M < r/2$ and, by property (iv) of the Christ decomposition, $Q_\alpha^M \cap B(z,r/2) = Q_\alpha^M$. So, using (14),

$$\begin{aligned}\mathcal{H}_N\big(\Xi^\bullet \cap B(x,r)\big) \ge \mathcal{H}_N\big(Q_\alpha^M \cap B(x,r)\big) \ge \mathcal{H}_N\big(Q_\alpha^M \cap B(z,r/2)\big) &= \mathcal{H}_N(Q_\alpha^M) \\ &\ge c(a_0\delta^M)^N \ge c(a_0\delta^M)^N r^N.\end{aligned}$$

Next, suppose $r \le 2c_1\delta^M$. Let $\ell \in \mathbb{N}_0$ be such that $2c_1\delta^\ell \le r \le 2c_1\delta^{\ell-1}$. Clearly, $\ell - 1 \ge M$. Choose $z \in B(x, c_1\delta^\ell)$ with $\alpha \in I_M$ such that $z \in Q_\alpha^M \subseteq \Xi^\bullet$ and $\beta \in I_\ell$ such that $z \in Q_\beta^\ell$ as in (b). We have $Q_\beta^\ell \subseteq Q_\alpha^M$ by property (ii) of the Christ decomposition.

As before, $B(z, r/2) \subseteq B(x, r)$ and from property (iv) of the Christ decomposition, and the choice of ℓ, we have $Q_\beta^\ell \cap B(z, r/2) = Q_\beta^\ell$. Using (14) and the choice of ℓ,

$$\begin{aligned}\mathcal{H}_N\big(\Xi^\bullet \cap B(x,r)\big) \geq \mathcal{H}_N\big(Q_\beta^\ell \cap B(x,r)\big) \geq \mathcal{H}_N\big(Q_\beta^\ell \cap B(z,r/2)\big) &= \mathcal{H}_N(Q_\beta^\ell) \\ &\geq c(a_0\delta^\ell)^N \geq c(a_0\delta/c_1)^N r^N.\end{aligned}$$

This completes the proof. □

Acknowledgment. We wish to thank Pertti Mattila (University of Helsinki) and Moritz Egert (Université Paris-Sud) for pointing out the Christ decomposition and ideas for the proof of Lemma 6.6. We are also grateful to the anonymous referee for their useful comments.

References

[1] H. Amann, *Linear and quasilinear parabolic problems. Vol. I*, Monographs in Mathematics, vol. 89, Birkhäuser Boston, Inc., Boston, MA, 1995, Abstract linear theory.

[2] W. Arendt, *Semigroups and evolution equations: functional calculus, regularity and kernel estimates*, Handbook of Differential Equations (C. M. Dafermos, E. Feireisl eds.), Elsevier/North Holland, 2004, pp. 1–85.

[3] M. Biegert, *On traces of Sobolev functions on the boundary of extension domains*, Proc. Amer. Math. Soc. **137** (2009), no. 12, 4169–4176.

[4] K. Brewster, D. Mitrea, I. Mitrea, and M. Mitrea, *Extending Sobolev functions with partially vanishing traces from locally (ε, δ)-domains and applications to mixed boundary problems*, J. Funct. Anal. **266** (2014), no. 7, 4314–4421.

[5] A. Carbonaro and O. Dragičević, *Convexity of power functions and bilinear embedding for divergence-form operators with complex coefficients*, J. Eur. Math. Soc. (2019), no. to appear.

[6] R. Chill, E. Fašangová, G. Metafune, and D. Pallara, *The sector of analyticity of the Ornstein-Uhlenbeck semigroup in L^p spaces with respect to invariant measure*, J. London Math. Soc. **71** (2005), 703–722.

[7] ———, *The sector of analyticity of nonsymmetric submarkovian semigroups generated by elliptic operators*, C. R. Acad. Sci. Paris **342** (2006), 909–914.

[8] M. Christ, *A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral*, Colloquium Mathematicae **60-61** (1990), no. 2, 601–628 (eng).

[9] A. Cialdea and V. Maz'ya, *Criterion for the L^p-dissipativity of second order differential operators with complex coefficients*, J. Math. Pures Appl. (9) **84** (2005), no. 8, 1067–1100.

[10] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645.

[11] D. Daners, *A priori estimates for solutions to elliptic equations on non-smooth domains*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), 793–813.

[12] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989.

[13] R. Denk, M. Hieber, and J. Prüss, *$\mathcal{R}$-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type*, Memoirs Amer. Math. Soc., vol. 166, Amer. Math. Soc., Providence, R.I., 2003.

[14] M. Egert, R. Haller-Dintelmann, and J. Rehberg, *Hardy's inequality for functions vanishing on a part of the boundary*, Potential Anal. **43** (2015), no. 1, 49–78.

[15] M. Egert and P. Tolksdorf, *Characterizations of Sobolev functions that vanish on a part of the boundary*, Discrete Contin. Dyn. Syst. Ser. S **10** (2017), no. 4, 729–743.

[16] J. A. Griepentrog, H.-C. Kaiser, and J. Rehberg, *Heat kernel and resolvent properties for second order elliptic differential operators with general boundary conditions on L^p*, Adv. Math. Sci. Appl. **11** (2001), no. 1, 87–112.

[17] P. Hajłasz, P. Koskela, and H. Tuominen, *Sobolev embeddings, extensions and measure density condition*, J. Funct. Anal. **254** (2008), no. 5, 1217–1234.

[18] A. Jonsson and H. Wallin, *Function spaces on subsets of $\mathbf{R}^n$*, Math. Rep. **2** (1984), no. 1, xiv+221.

[19] N. Kalton and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann. **321** (2001), 319–345.

[20] P. C. Kunstmann, *L_p-spectral properties of the Neumann Laplacian on horns, comets and stars*, Math. Z. **242** (2002), 183–201.

[21] P. C. Kunstmann and L. Weis, *Maximal L^p regularity for parabolic equations, Fourier multiplier theorems and H^∞ functional calculus*, Levico Lectures, Proceedings of the Autumn School on Evolution Equations and Semigroups (M. Iannelli, R. Nagel, S. Piazzera eds.), vol. 69, Springer Verlag, Heidelberg, Berlin, 2004, pp. 65–320.

[22] Damien Lamberton, *Équations d'évolution linéaires associées à des semi-groupes de contractions dans les espaces L^p*, J. Funct. Anal. **72** (1987), no. 2, 252–262.

[23] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and Their Applications, vol. 16, Birkhäuser, Basel, 1995.

[24] V. Maz'ya, *Sobolev spaces with applications to elliptic partial differential equations*, augmented ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 342, Springer, Heidelberg, 2011.

[25] V. G. Maz'ya and S. V. Poborchiĭ, *Theorems for embedding Sobolev spaces on domains with a peak and on Hölder domains*, Algebra i Analiz **18** (2006), no. 4, 95–126.

[26] A. J. Morris, *The Kato square root problem on submanifolds*, J. London Math. Soc. **86** (2012), no. 3, 879–910.

[27] E. M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Mathematical Society Monographs, vol. 30, Princeton University Press, Princeton, 2004.

[28] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Berlin, 1983.

[29] J. Prüss and G. Simonett, *Moving interfaces and quasilinear parabolic evolution equations*, Monographs in Mathematics, vol. 105, Birkhäuser/Springer, [Cham], 2016.

[30] A. F. M. ter Elst, R. Haller-Dintelmann, J. Rehberg, and P. Tolksdorf, *On the L^p-theory for second-order elliptic operators in divergence form with complex coefficients*, Preprint (2019), https://arxiv.org/abs/1903.06692.

[31] A. F. M. ter Elst, M. Meyries, and J. Rehberg, *Parabolic equations with dynamical boundary conditions and source terms on interfaces*, Ann. Mat. Pura Appl. (4) **193** (2014), no. 5, 1295–1318.

[32] H. Triebel, *A note on function spaces in rough domains*, Tr. Mat. Inst. Steklova **293** (2016), no. Funktsional. Prostranstva, Teor. Priblizh., Smezhnye Razdely Mat. Anal., 346–351, English version published in Proc. Steklov Inst. Math. **293** (2016), no. 1, 338–342.

[33] A. Ukhlov, *Extension operators on Sobolev spaces with decreasing integrability*, Preprint (2019), https://arxiv.org/abs/1908.09322.