Annali di Matematica Pura ed Applicata manuscript No.
(will be inserted by the editor)

Higher regularity for solutions to elliptic systems in divergence form subject to mixed boundary conditions

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Received: date / Accepted: date

Abstract This note combines some known results from operator- and interpolation theory to show that elliptic systems in divergence form admit maximal elliptic regularity on the Bessel potential scale $\mathrm{H}^{s-1}_D(\Omega)$ for $s > 0$ sufficiently small, if the coefficient in the main part satisfies a certain multiplier property on the spaces $\mathrm{H}^s(\Omega)$. Ellipticity is enforced by assuming a Gårding inequality and the result is established for spaces incorporating mixed boundary conditions with very low regularity requirements for the underlying spatial set. To illustrate the applicability of our results, two examples are provided. Firstly, a phase-field damage model is given as a practical application where higher differentiability results are obtained as a corollary to our findings and needed to show an improved numerical approximation rate. Secondly, it is shown how the maximal elliptic regularity result can be used in the context of quasilinear parabolic equations.

Mathematics Subject Classification (2010) Primary: 35B65, Secondary: 35J57, 35J25

Keywords maximal elliptic regularity, non Lipschitz coefficients, second order divergence operators, elliptic system, mixed boundary conditions, phase-field damage

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1 Introduction

In this paper, we consider the weak form of a system of elliptic $(n\times n)$ operators $\mathbb{A}$ in divergence form induced by the tensor A for a function $u\colon \Omega \to \mathbb{C}^n$ (or $\mathbb{R}^n$):

$$\begin{aligned} (\mathbb{A}u)_1 &= -\sum_{\alpha,\beta=1}^{d}\sum_{j=1}^{n}\partial_\alpha\bigl(A^{1,j}_{\alpha,\beta}\partial_\beta u_j\bigr) \\ \vdots \quad &\vdots \qquad\qquad \vdots \\ (\mathbb{A}u)_n &= -\sum_{\alpha,\beta=1}^{d}\sum_{j=1}^{n}\partial_\alpha\bigl(A^{n,j}_{\alpha,\beta}\partial_\beta u_j\bigr). \end{aligned} \tag{1}$$

Each equation, for $j \in \{1,\dots,n\}$, in this system is subject to possibly mixed boundary conditions with a Dirichlet boundary part D_j and we set $D := \times_{j=1}^{n} D_j$. Precise assumptions on A, the domain Ω, and the Dirichlet boundary parts D_j are given below in Assumptions 1 and 2.

As a motivation for our results, let us assume for the moment that the operator $\mathbb{A}$ defines an isomorphism between the Sobolev Hilbert spaces $\mathbb{H}^1_D(\Omega) \to \mathbb{H}^{-1}_D(\Omega)$, as it can be asserted by the notorious Lax-Milgram theorem in many cases. It is known under mild assumptions on Ω and D that if $A^{i,j}_{\alpha,\beta} \in \mathrm{L}^\infty(\Omega)$, then $\mathbb{A}$ remains an isomorphism for small perturbations in the integrability scale, i.e., there exists $\varepsilon > 0$ such that the mapping

$$\mathbb{A}\colon \mathbb{H}^{1,p}_D(\Omega) \to \mathbb{H}^{-1,p}_D(\Omega)$$

remains an isomorphism for all p satisfying $2-\varepsilon \le p \le 2+\varepsilon$, see, e.g., [12,13] or [14].

We will show that such a result is also true on the *differentiability* scale: Indeed, we prove that if A is a multiplier on $\mathbb{H}^\varepsilon$ for some $0 < \varepsilon < \frac{1}{2}$, then there exists $\vartheta > 0$ such that the mapping

$$\mathbb{A}\colon \mathbb{H}^{1+\theta}_D(\Omega) \to \mathbb{H}^{\theta-1}_D(\Omega)$$

is still an isomorphism for any θ satisfying $-\vartheta \le \theta \le \vartheta$. Moreover, the norm of the inverse of $\mathbb{A}$ is uniform in its coercivity constant and the bound on the multiplier norm; in particular, it does not depend explicitly on the actual multiplier at hand. Similar results have been obtained by Jochmann in [15] for the case of a scalar elliptic problem on a domain with a piecewise smooth boundary. Our work can therefore be seen as an extension of these results to elliptic systems, while allowing much less regular geometries for Ω and the boundary parts D_j. We note that the required multiplier property for A is in particular satisfied if A is σ-Hölder-continuous for $\sigma > \varepsilon$ or if A is a characteristic function of an open subset of Ω with locally finite perimeter.

Such results are interesting, firstly because they provide a sharp maximal elliptic regularity result for the abstract equation $\mathbb{A}u = f$ in $\mathbb{H}^{\theta-1}_D(\Omega)$. This allows for instance to obtain certain improved convergence rates in the analysis of $\mathbb{H}^1$-discretization errors for such equations, which is possible only if a gap in differentiability is present, see our example in Section 7. Further, the fact that the norm of the inverse of $\mathbb{A}$ is uniform for all multipliers with a certain coercivity constant and multiplier norm makes the result also attractive to use in a nonlinear elliptic setting via fixed-point techniques. Finally, the knowledge that $\mathbb{H}^{1+\theta}_D(\Omega)$ is the domain of $\mathbb{A}$ over $\mathbb{H}^{\theta-1}_D(\Omega)$ for *all* coefficient functions in a certain class is a very valuable information when aiming to treat nonautonomous time-dependent problems of the form

$$u'(t) + \mathbb{A}(t)u(t) = f(t) \quad \text{in } \mathbb{H}^{\theta-1}_D(\Omega), \quad u(0) = u_0,$$

see, e.g., [1,22], or even for quasilinear equations such as

$$u'(t) + \mathbb{A}(u(t))u(t) = F(u(t)) \quad \text{in } \mathbb{H}^{\theta-1}_D(\Omega), \quad u(0) = u_0.$$

We explain the latter in some more detail in Section 8.

Throughout the paper, the considered Banach spaces are in general complex vector spaces. By $\cong$ we understand that two normed spaces are equal up to equivalent norms. Moreover, the restriction of $f\colon U \to \mathbb{C}$ to $\Lambda \subseteq U$ will be denoted by $f_{\restriction \Lambda}$ and we use $B_r(\mathrm{x})$ for the ball of radius r around x in $\mathbb{R}^d$.

The rest of the paper is structured as follows: We will start by stating our main result in Section 2 together with the notation of the subsequent sections. In Section 3, we will give the details on the assumed regularity of the domain: we assume that (the closure of) the non-Dirichlet boundary parts admit bi-Lipschitz boundary charts and allow the Dirichlet parts of the domain to be $(d-1)$-sets. In Section 4, we will define the Bessel potential function spaces needed in the statement of our result. The collection of preliminaries ends in Section 5, where we briefly introduce the concept of a multiplier space and provide some more accessible examples for when a coefficient function is in fact a multiplier. After these preparations, we come to the proof of the main result in Section 6. We conclude the paper by applications of our results to a phase-field fracture/damage model in Section 7 and to quasilinear equations in Section 8.

2 Main result

We first give our main result. All occurring spaces and the notion of a multiplier are formally introduced and defined below (cf. Definitions 2, 3 and 4).

Assumption 1 For $i, j \in \{1, \dots, n\}$, each matrix $A^{i,j}$ is a $(d \times d)$ matrix with $A^{i,j}_{\alpha,\beta} \in \mathrm{L}^\infty(\Omega)$ for $\alpha, \beta \in \{1, \dots, d\}$.

To formulate the weak form of the elliptic system operator (1), let

$$\mathbb{H}^1_D(\Omega) := \prod_{j=1}^{n} \mathrm{H}^1_{D_j}(\Omega),$$

and let $\mathbb{H}^{-1}_D(\Omega)$ be the anti-dual space of $\mathbb{H}^1_D(\Omega)$. For a tensor A satisfying Assumption 1, we define the form $a\colon \mathbb{H}^1_D(\Omega)\times\mathbb{H}^1_D(\Omega)\to\mathbb{C}$ and the divergence-gradient system operator $-\nabla\cdot A\nabla\colon \mathbb{H}^1_D(\Omega)\to\mathbb{H}^{-1}_D(\Omega)$ by

$$\begin{aligned}\big\langle -\nabla\cdot A\nabla u, v\big\rangle &:= a(u,v)\\ &:= \sum_{i,j=1}^{n}\int_\Omega \big(A^{i,j}\nabla u_j\big)\cdot\nabla\overline{v_i}\,\mathrm{dx}\quad \text{for } u,v\in\mathbb{H}^1_D(\Omega).\end{aligned}\tag{2}$$

We extend this slightly by defining $-\nabla\cdot A\nabla+\gamma\colon \mathbb{H}^1_D(\Omega)\to\mathbb{H}^{-1}_D(\Omega)$ for $\gamma\ge 0$ by

$$\big\langle\big(-\nabla\cdot A\nabla+\gamma\big)u,v\big\rangle := \big\langle -\nabla\cdot A\nabla u,v\big\rangle + \sum_{j=1}^{n}\int_\Omega \gamma u_j\overline{v_j}\,\mathrm{dx}$$

and formulate our main result as follows:

Theorem 1 *Let Assumptions 1 and 2 be satisfied and suppose that the system* (1) *is elliptic in the sense that it satisfies a* Gårding inequality, *i.e., there exist* $\lambda>0$ *and* $\mu\ge 0$ *such that*

$$\operatorname{Re}\big(a(u,u)\big)\ge\sum_{i=1}^{n}\lambda\|\nabla u_i\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^d)}-\mu\|u_i\|^2_{\mathrm{L}^2(\Omega)}\quad \textit{for all } u\in\mathbb{H}^1_D(\Omega).$$

Assume further that each matrix $A^{i,j}$ *is a multiplier on* $\mathrm{H}^\varepsilon(\Omega)^d$ *for some* $0<\varepsilon<\frac{1}{2}$. *Then there exist* $\gamma\ge 0$ *and* $0<\delta\le\varepsilon$ *such that*

$$-\nabla\cdot A\nabla+\gamma\in\mathcal{L}_{\mathrm{iso}}\big(\mathbb{H}^{\theta+1}_D(\Omega);\mathbb{H}^{\theta-1}_D(\Omega)\big)\quad \textit{for all } |\theta|<\delta,\tag{3}$$

i.e., $-\nabla\cdot A\nabla+\gamma$ *is a topological isomorphism between* $\mathbb{H}^{\theta+1}_D(\Omega)$ *and* $\mathbb{H}^{\theta-1}_D(\Omega)$ *for every* $-\delta<\theta<\delta$. *Both the size of* δ *and the norm of the inverse of* $-\nabla\cdot A\nabla+\gamma$ *are uniformly bounded w.r.t. the multiplier norm of* A, *the coefficients in the Gårding inequality and* γ.

Remark 1
(i) The need for the perturbation $\gamma\ge 0$ in Theorem 1 is due to the possibility that 0 might be an eigenvalue of $\mathbb{A}$. If this is not the case, $\gamma=0$ can be chosen. In particular, $\gamma=0$ is allowed if $\mu=0$ and if a Poincaré inequality holds true for $\mathbb{H}^1_D(\Omega)$. The latter is already satisfied for $D\ne\emptyset$ in our geometric setting as given in Section 3 below, cf. [3, Rem. 3.4].
(ii) We give sufficient conditions for the matrix functions $A^{i,j}$ to be multipliers on $\mathrm{H}^\varepsilon(\Omega)^d$ in Lemma 1 below. A particular case is when $A^{i,j}_{\alpha\beta}\in\mathrm{C}^\sigma(\Omega)$ for some $\varepsilon<\sigma<1$ and all $\alpha,\beta\in\{1,\dots,d\}$, where $\mathrm{C}^\sigma(\Omega)$ is the space of σ-Hölder continuous functions on Ω. This also implies that $\mathrm{C}^{\frac{1}{2}}(\Omega)$ is always a suitable multiplier space for Theorem 1. Note however that a multiplier need not necessarily be continuous, see Remark 5.

(iii) We consider the Gårding inequality as the adequate abstract tool to enforce coercivity in our context since it is known that if A satisfies the *Legendre-Hadamard condition* and the coefficient functions are uniformly continuous (cf. the previous point), then the Gårding inequality is indeed satisfied at least for $D = \emptyset$ (see [8, Ch. 3.4.3]). Coercivity of system operators $-\nabla \cdot A\nabla$ in the setting $D \neq \emptyset$ without a very strong ellipticity assumption in the form of a *Legendre condition* is both an interesting and (very) difficult topic, see, e.g., [24,28] and the references therein.

Theorem 1 yields the following corollary:

Corollary 1 *In the situation of Theorem 1, let* $f \in \mathbb{H}^{\theta-1}_D(\Omega)$ *for some* $0 < \theta < \delta$. *Then the elliptic system*

$$-\nabla \cdot A\nabla u + \gamma u = f \quad \text{in } \mathbb{H}^{\theta-1}_D(\Omega) \tag{4}$$

has a unique solution $u \in \mathbb{H}^{\theta+1}_D(\Omega)$ *satisfying*

$$\|u\|_{\mathbb{H}^{\theta+1}_D(\Omega)} \le C\|f\|_{\mathbb{H}^{\theta-1}_D(\Omega)}$$

for some constant $C \ge 0$ *independent of* f *and uniform in the multiplier norm of* A, *the constants in the Gårding inequality and* γ. *Moreover, for all* $0 < \eta < \theta$ *there exist* $p > 2$ *and* $C^\bullet \ge 0$ *such that* $u \in \mathbb{H}^{1+\eta,p}_D(\Omega)$ *and*

$$\|u\|_{\mathbb{H}^{1+\eta,p}_D(\Omega)} \le C^\bullet\|f\|_{\mathbb{H}^{\theta-1}_D(\Omega)}$$

where $C^\bullet$ *is uniform in the same quantities as* C *is.*

Remark 2 A particular case for $f = (f_1, \ldots, f_n)$ being in $\mathbb{H}^{\theta-1}_D(\Omega)$ is when we have $f_j \in \mathrm{L}^{q_j}(\Omega)$ for $q_j \ge \frac{2d}{d+2(1-\theta)}$. This follows from the embedding

$$\mathrm{L}^{q_j}(\Omega) \hookrightarrow \mathrm{H}^{\theta-1}_{D_j}(\Omega) \quad \text{if } q_j \ge \frac{2d}{d+2(1-\theta)}$$

which can be inferred from the extension property for $\mathrm{H}^{1-\theta}_{D_j}(\Omega)$ as in [7, Thm. 5.1], the embedding $\mathrm{H}^{1-\theta}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^{q_j'}(\mathbb{R}^n)$ in $\mathbb{R}^n$ [25, Ch. 2.8.1], and a duality argument; see Section 4 for the formal definitions of the function spaces.

3 Assumptions on the domain

We formulate the assumptions on the spatial domain $\Omega \subset \mathbb{R}^d$ and its boundary. As part of the assumptions on Theorem 1, these are supposed to be valid in all of the following. A preliminary definition we need is the following:

Definition 1 **($(d-1)$-set)** Let $F \subset \mathbb{R}^d$ be a Borel set. We say that F is a *$(d-1)$-set* or that F satisfies the *Ahlfors-David condition* if there is $c \geq 1$ such that

$$c^{-1} r^{d-1} \leq \mathcal{H}^{d-1}\big(F \cap B_r(\mathrm{x})\big) \leq c r^{d-1} \quad \text{for all } \mathrm{x} \in F,\ 0 < r \leq 1,$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure.

The assumptions on Ω and D_j for $j \in \{1, \dots, n\}$ are then as follows, where we set $\mathfrak{D} := \bigcap_{j=1}^n D_j$:

Assumption 2 The set $\Omega \subset \mathbb{R}^d$ is a bounded domain and for each $j \in \{1, \dots, n\}$, the set $D_j \subseteq \partial\Omega$ is either empty or a closed $(d-1)$-set. For every point $\mathrm{x} \in \overline{\partial\Omega \setminus \mathfrak{D}}$ there are Lipschitz boundary charts available, that is, there exists an open neighborhood U_x of x and a bi-Lipschitz map $\phi_\mathrm{x} \colon U_\mathrm{x} \to (-1,1)^d$ such that $\phi_\mathrm{x}(\mathrm{x}) = 0$ and

$$\phi_\mathrm{x}(U_\mathrm{x} \cap \Omega) = \big\{\mathrm{x} \in (-1,1)^d \colon \mathrm{x}_d < 0\big\},$$
$$\phi_\mathrm{x}(U_\mathrm{x} \cap \partial\Omega) = \big\{\mathrm{x} \in (-1,1)^d \colon \mathrm{x}_d = 0\big\}.$$

Remark 3 (i) For $\mathfrak{D} = \emptyset$, the assumptions on Ω fall back to that of a classical Lipschitz domain (cf. [11]). On the other side of the spectrum, for $\mathfrak{D} = \partial\Omega$, so pure Dirichlet conditions for every equation in the system (1), we do not require local descriptions of $\partial\Omega$ by boundary charts *at all*.
(ii) If $\Omega \cup D_j$ is regular in the sense of Gröger (cf. [12,13]) for some $j \in \{1, \dots, n\}$, then Assumption 2 is already satisfied. Indeed, in this case D_j is already a $(d-1)$-set, and there are already bi-Lipschitz charts available for the *whole* $\partial\Omega$, so Ω is again a Lipschitz domain. This follows from the facts that the concept of Gröger requires that $D_j \supseteq \mathfrak{D}$ is also described by local bi-Lipschitz charts as $\overline{\partial\Omega \setminus \mathfrak{D}}$ is in Assumption 2, that such a local bi-Lipschitz description of D_j implies that D_j is a $(d-1)$-set by [16, Ch. II.1.1, Ex. 1], and that finite unions of $(d-1)$-sets are again $(d-1)$-sets. Clearly, Assumption 2 is also satisfied if $\Omega \cup D_j$ is regular in the sense of Gröger for *every* $j \in \{1, \dots, n\}$.
(iii) With the same argument as in the previous point, we find that under Assumption 2, the whole boundary $\partial\Omega$ is always a $(d-1)$-set.

4 Definitions and basics

We move to exact definitions of the fundamental function spaces. Here, we mostly work only with the scalar-valued spaces $\mathrm{H}^{s,p}_F(\Omega)$ for $(d-1)$-sets F satisfying $\mathfrak{D} \subseteq F \subseteq \partial\Omega$ since their properties translate to n-fold products of such spaces immediately. Note that under Assumption 2, every D_j is a valid choice for such F, as is $\partial\Omega$ by Remark 3 iii.

Definition 2 (Bessel potential spaces) Let $1 < p < \infty$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$, and consider a $(d-1)$-set F such that $\mathfrak{D} \subseteq F \subseteq \partial\Omega$. Denote by $\mathrm{H}^{s,p}(\mathbb{R}^d)$ the classical Bessel potential spaces, cf. [25, Ch. 2.3.1/Thm. 2.3.3]. Then we make the following definitions:

(i) Set

$$\mathrm{H}^{s,p}_F(\mathbb{R}^d) := \Big\{ f \in \mathrm{H}^{s,p}(\mathbb{R}^d) \colon \lim_{r \searrow 0} \frac{1}{|B_r(\mathrm{x})|} \int_{B_r(\mathrm{x})} f(\mathrm{y}) \,\mathrm{dy} = 0 \text{ for } \mathcal{H}^{d-1}\text{-a.e. } \mathrm{x} \in F \Big\}$$

with $\mathrm{H}^s_F(\mathbb{R}^d) := \mathrm{H}^{s,2}_F(\mathbb{R}^d)$ and $\|\cdot\|_{\mathrm{H}^{s,p}_F(\mathbb{R}^d)} = \|\cdot\|_{\mathrm{H}^{s,p}(\mathbb{R}^d)}$.

(ii) Further, set $\mathrm{H}^{s,p}_F(\Omega) := \big\{ f_{\restriction\Omega} \colon f \in \mathrm{H}^{s,p}_F(\mathbb{R}^d) \big\}$, equipped with the factor space norm

$$\|f\|_{\mathrm{H}^{s,p}_F(\Omega)} := \inf\big\{ \|g\|_{\mathrm{H}^{s,p}(\mathbb{R}^d)} \colon g \in \mathrm{H}^{s,p}_F(\mathbb{R}^d),\ g_{\restriction\Omega} = f \big\}.$$

We set, again, $\mathrm{H}^s_F(\Omega) := \mathrm{H}^{s,2}_F(\Omega)$, and for $F = \emptyset$, we write $\mathrm{H}^{s,p}(\Omega) := \mathrm{H}^{s,p}_\emptyset(\Omega)$.

(iii) Denote by $\mathrm{H}^{-s}_F(\mathbb{R}^d)$ and $\mathrm{H}^{-s}_F(\Omega)$ the space of antilinear continuous functionals acting on $\mathrm{H}^s_F(\mathbb{R}^d)$ and $\mathrm{H}^s_F(\Omega)$, respectively. We agree that the convention $\mathrm{H}^{-s}(\Omega) := \mathrm{H}^{-s}_\emptyset(\Omega)$ still applies.

(iv) For $\Lambda \in \{\Omega, \mathbb{R}^d\}$ and D_j from Assumption 2, set $\mathbb{H}^{s,p}_D(\Lambda) := \prod_{j=1}^n \mathrm{H}^{s,p}_{D_j}(\Lambda)$, with all the previous conventions for $p = 2$, and let $\mathbb{H}^{-s}_D(\Lambda)$ be the space of continuous antilinear functionals on $\mathbb{H}^s_D(\Lambda)$, so $\mathbb{H}^{-s}_D(\Lambda) := \prod_{j=1}^n \mathrm{H}^{-s}_{D_j}(\Lambda)$.

Remark 4 (i) For $1 \le s < 1 + \frac{1}{p}$, it is easy to see that $\mathrm{H}^{s,p}_F(\mathbb{R}^d) = \mathrm{H}^{1,p}_F(\mathbb{R}^d) \cap \mathrm{H}^{s,p}(\mathbb{R}^d)$ and that accordingly $\mathrm{H}^{s,p}_F(\Omega) \subseteq \mathrm{H}^{1,p}_F(\Omega) \cap \mathrm{H}^{s,p}(\Omega)$. If there exists an extension operator E which maps $\mathrm{H}^{1,p}_F(\Omega)$ into $\mathrm{H}^{1,p}_F(\mathbb{R}^d)$ and $\mathrm{H}^{s,p}(\Omega)$ into $\mathrm{H}^{s,p}(\mathbb{R}^d)$ at the same time such that $Ef_{\restriction\Omega} = f$, then the reverse inclusion and thus

$$\mathrm{H}^{s,p}_F(\Omega) = \mathrm{H}^{1,p}_F(\Omega) \cap \mathrm{H}^{s,p}(\Omega)$$

follows. A particular case in which this extension property for Ω is satisfied is when $\Omega \cup D_j$ is regular in the sense of Gröger for some $j \in \{1, \ldots, n\}$ (cf. Remark 3 (ii)) because then Ω is a Lipschitz domain for which the $\mathrm{H}^{s,p}$-extension property is classical ([9, Thm. 7.25]), and the preservation of the zero trace on F for the $\mathrm{H}^{1,p}$-extension follows as in [6, Cor. 2.2.13].

(ii) Many authors commonly use $\mathrm{H}^s_0(\Omega)$ instead of $\mathrm{H}^s_{\partial\Omega}(\Omega)$ and $\mathrm{H}^{-1}(\Omega)$ instead of $\mathrm{H}^{-1}_{\partial\Omega}(\Omega)$. We feel that while this is adequate as long as only one fixed part of the boundary, e.g. $F = \partial\Omega$, is considered, a more careful notation is needed in view of the importance of both the sets D_j *and* $\partial\Omega$.

The rather abstract definition of $\mathrm{H}^1_F(\Omega)$ turns out to be equivalent to the classical Sobolev space with partially vanishing trace $\mathrm{W}^{1,2}_F(\Omega)$ which we formally define as follows.

Definition 3 (Sobolev spaces with partially vanishing trace) Let F be a $(d-1)$-set satisfying $\mathfrak{D} \subseteq F \subseteq \partial\Omega$ and let $\Lambda \subseteq \mathbb{R}^d$ be a domain. Then we set

$$\mathrm{C}^\infty_F(\Lambda) := \left\{ f_{\restriction \Lambda} \colon f \in \mathrm{C}^\infty_c(\mathbb{R}^d),\ \operatorname{supp} f \cap F = \emptyset \right\}$$

and

$$\mathrm{W}^{1,2}_F(\Lambda) := \overline{\mathrm{C}^\infty_F(\Lambda)}^{\|\cdot\|_{\mathrm{W}^{1,2}(\Lambda)}}$$

for

$$\|f\|_{\mathrm{W}^{1,2}(\Lambda)} := \left(\int_\Lambda |f|^2 + \|\nabla f\|_2^2 \,\mathrm{dx} \right)^{\frac{1}{2}}.$$

Proposition 1 ([7, Cor. 3.8]) *Let F be a $(d-1)$-set satisfying $\mathfrak{D} \subseteq F \subseteq \partial\Omega$. Then there holds $\mathrm{W}^{1,2}_F(\Omega) \cong \mathrm{H}^1_F(\Omega)$.*

Using Proposition 1, we easily verify that $-\nabla \cdot A\nabla$ as in (2) is indeed well defined as an operator from $\mathbb{H}^1_D(\Omega)$ to $\mathbb{H}^{-1}_D(\Omega)$ under Assumption 1.

5 Multipliers

We finally turn to the notion of a multiplier. In the present case, the following definition is sufficient:

Definition 4 (Multiplier) Let X and Y be Banach spaces whose elements are functions on a common domain of definition Λ. We say that Y is a *multiplier space* of X if for every $\rho \in Y$ the pointwise multiplication operator T_ρ defined by $(T_\rho f)(\mathrm{x}) := \rho(\mathrm{x}) f(\mathrm{x})$ for $\mathrm{x} \in \Lambda$ is a continuous linear operator from X into itself. In this case, the functions $\rho \in Y$ are called *multipliers* for X.

We give sufficient conditions on when a matrix function is in fact a multiplier on spaces of the type $\mathrm{H}^\varepsilon(\Omega)^d$ for $0 < \varepsilon < \frac{1}{2}$, as required in Theorem 1. We do so using Besov spaces $\mathrm{B}^s_{p,q}(\Omega)$ including those of non-standard type $p = \infty$. For $0 < s < 1$ and $q = \infty$, the latter coincide with the Hölder spaces, see [26]. See also [25] and or [23] for the definitions of the Besov space and more.

Clearly, for a matrix-valued function $S \colon \Omega \to \mathbb{C}^{d\times d}$ to be a multiplier on $\mathrm{H}^\varepsilon(\Omega)^d$ it is sufficient if every component function $S_{\alpha,\beta}$ for $\alpha, \beta \in \{1, \dots, d\}$ is a multiplier on $\mathrm{H}^\varepsilon(\Omega)$ alone. We thus just give conditions for this basing on [23] and [26].

Lemma 1 *Let $0 < \varepsilon < \frac{1}{2}$ be given. Then $\mathrm{B}^\varepsilon_{d(1+\eta)/\varepsilon,2}(\Omega)$ and $\mathrm{C}^\sigma(\Omega)$ are multiplier spaces for $\mathrm{H}^\varepsilon(\Omega)$ for every $0 < \eta \le \infty$ and $\varepsilon < \sigma < 1$.*

Proof The results from [23] and [26] in the following proof are originally stated only for function spaces on $\mathbb{R}^d$. The occurring function spaces on Ω are defined as the restrictions to Ω of the ones on $\mathbb{R}^d$ (cf. Definition 2) which however allows to transfer the results from $\mathbb{R}^d$ to Ω by considering functions in the function spaces on $\mathbb{R}^d$ whose restriction is the function of interest defined on Ω. For the Hölder spaces, recall that there is the McShane-Whitney extension operator [19].

The multiplier property for $\mathrm{B}^{\varepsilon}_{d(1+\eta)/\varepsilon,2}(\Omega)$ for $0 < \eta \leq \infty$ is proven in [23, Thm. 2, Ch. 4.4.4]. Please note here that $\mathrm{H}^s(\Omega) = \mathrm{F}^s_{2,2}(\Omega) = \mathrm{B}^s_{2,2}(\Omega)$. For the case $\eta = \infty$, so $\mathrm{B}^{\varepsilon}_{\infty,2}(\Omega)$ as a multiplier space, see also [23, Ch. 4.7.1]. The assertion for the Hölder spaces now follows from the embedding

$$\mathrm{B}^{\sigma}_{\infty,\infty}(\Omega) \hookrightarrow \mathrm{B}^{\varepsilon}_{\infty,2}(\Omega) \quad \text{for } \sigma > \varepsilon \tag{5}$$

together with [26, Thm. 4] from which we have

$$\mathrm{C}^{\sigma}(\Omega) \cong \mathrm{B}^{\sigma}_{\infty,\infty}(\Omega) \quad \text{for } 0 < \sigma < 1.$$

Note that the embedding in (5) is not explicitly stated in [26], but follows immediately from the definition of the Besov space there, see [26, p. 77/78], cf. also [23, Ch. 2.2.1].

See also [15, Lem. 2] for a similar multiplier result involving σ-Hölder functions for $\sigma > \varepsilon$.

Remark 5 Let us point out once more that the conditions in Lemma 1 are merely *sufficient* and in no way necessary. In fact, due to the embeddings

$$\mathrm{B}^{\varepsilon}_{d(1+\eta)/\varepsilon,2}(\Omega) \hookrightarrow \mathrm{C}^{\frac{\varepsilon\eta}{1+\eta}}(\Omega) \quad \text{and} \quad \mathrm{B}^{\varepsilon}_{\infty,2}(\Omega) \hookrightarrow \mathrm{B}^{\varepsilon}_{\infty,\infty}(\Omega) \cong \mathrm{C}^{\varepsilon}(\Omega),$$

we see that the given multipliers are all at least Hölder continuous, and $\mathrm{B}^{\varepsilon}_{\infty,2}(\Omega)$ lies in fact between $\mathrm{C}^{\sigma}(\Omega)$ and $\mathrm{C}^{\varepsilon}(\Omega)$ for any $0 < \varepsilon < \sigma$, recall (5). But it is also known that the—clearly discontinuous—characteristic function χ_{Ξ} for an open set $\Xi \subset \Omega$ with locally finite perimeter is also a multiplier on $\mathrm{H}^{\varepsilon}(\Omega)$ whenever $|\varepsilon| < \frac{1}{2}$, see [23, p. 214ff]. A general intrinsic characterization of multipliers on $\mathrm{H}^{\varepsilon}(\Omega)$ in terms of usual function spaces seems not to be available. We refer to [23,18] and related works, see also [17, Sect. 5].

6 Proof of the main results

The proof of Theorem 1 rests on the following fundamental theorem by Šneĭberg [27], cf. also [6, Ch. 1.3.5]. For the notions from interpolation theory we refer to [25, Ch. 1.2, 1.9].

Theorem 2 (Šneĭberg stability theorem) *Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples of Banach spaces and let T be a continuous linear operator compatible with that interpolation couple. Then the set*

$$\Big\{\theta \in (0,1) \colon T \in \mathcal{L}_{\mathrm{iso}}\big([X_0, X_1]_\theta; [Y_0, Y_1]_\theta\big)\Big\} \tag{6}$$

is open.

Remark 6 Given a number ϑ which is an element of the set (6) in Theorem 2, there exist estimates on the size of the open set (6), see [6, Ch. 1.3.5]. These show that the size depends on the operator norms of T as a linear operator from X_i to Y_i for $i = 1, 2$, and the operator norm of T^{-1} between $[Y_0, Y_1]_\vartheta$ and $[X_0, X_1]_\vartheta$. This is in fact the connection to the claim about uniformity of δ and the norm of the inverses of $-\nabla \cdot A\nabla + \gamma$ in the main results Theorem 1 and Corollary 1.

In order to use Theorem 2 we need to have a suitable interpolation scale at hand. For this, we rely on [7, Ch. 7] from which we cite

Theorem 3 ([7, Thm. 7.1]) *Let F be a $(d-1)$-set satisfying $\mathfrak{D} \subseteq F \subseteq \partial\Omega$. Let further $0 < \theta < 1$ and $\frac{1}{2} < s_0, s_1 < \frac{3}{2}$ and put $s_\theta := (1-\theta)s_0 + \theta s_1$. Then*

$$\big[\mathrm{H}^{s_0}_F(\Omega), \mathrm{H}^{s_1}_F(\Omega)\big]_\theta = \mathrm{H}^{s_\theta}_F(\Omega)$$

and

$$\big[\mathrm{L}^2(\Omega), \mathrm{H}^1_F(\Omega)\big]_\theta = \begin{cases} \mathrm{H}^\theta_F(\Omega) & \text{if } \theta > \frac{1}{2}, \\ \mathrm{H}^\theta(\Omega) & \text{if } \theta < \frac{1}{2}. \end{cases}$$

Before we prove our main result, we establish a few preparatory lemmas building upon Theorem 3.

Lemma 2 *In the situation of Theorem 3, we also have*

$$\big[\mathrm{H}^{-s_0}_F(\Omega), \mathrm{H}^{-s_1}_F(\Omega)\big]_\theta = \mathrm{H}^{-s_\theta}_F(\Omega)$$

and

$$\big[\mathrm{L}^2(\Omega), \mathrm{H}^{-1}_F(\Omega)\big]_\theta = \begin{cases} \mathrm{H}^{-\theta}_F(\Omega) & \text{if } \theta > \frac{1}{2}, \\ \mathrm{H}^{-\theta}(\Omega) & \text{if } \theta < \frac{1}{2}. \end{cases}$$

Proof This follows quite immediately from the result that the $\mathrm{H}^s_F(\Omega)$ spaces are reflexive [7, Cor. 5.3] and general interpolation duality properties [25, Ch. 1.11.3]. Here, density of $\mathrm{H}^{s_0}_F(\Omega) \cap \mathrm{H}^{s_1}_F(\Omega) = \mathrm{H}^{\max(s_0,s_1)}_F(\Omega)$ in $\mathrm{H}^{s_0}_F(\Omega)$ and $\mathrm{H}^{s_1}_F(\Omega)$ follows from density of $\mathrm{H}^{\max(s_0,s_1)}(\mathbb{R}^d)$ in $\mathrm{H}^{s_0}(\mathbb{R}^d)$ and $\mathrm{H}^{s_1}(\mathbb{R}^d)$ and the characterization $\mathrm{H}^s_F(\mathbb{R}^d) = P_F \mathrm{H}^s(\mathbb{R}^d)$ for a bounded linear projection P_F as proven in [7, Cor. 3.5].

Now it only remains to set the stage for the extension of $-\nabla \cdot A\nabla$ to $\mathbb{H}^s_D(\Omega)$ for $s \neq 1$ before we can give the proof of the main results.

Lemma 3 *Let F be a $(d-1)$-set satisfying $\mathfrak{D} \subseteq F \subseteq \partial\Omega$ and let $0 < \sigma < \frac{1}{2}$. Then the weak gradient $\nabla \in \mathcal{L}(\mathrm{H}^1_F(\Omega); \mathrm{L}^2(\Omega)^d)$ maps $\mathrm{H}^{\sigma+1}_F(\Omega)$ continuously into $\mathrm{H}^\sigma(\Omega)^d$ and admits a unique continuous linear extension $\nabla \colon \mathrm{H}^{1-\sigma}_F(\Omega) \to \mathrm{H}^{-\sigma}(\Omega)^d$.*

Proof The first assertion follows from the corresponding property of $\mathrm{H}^{\sigma+1}(\mathbb{R}^d)$ and the definition of the $\mathrm{H}^{\sigma+1}_F(\Omega)$ spaces. For the second assertion, observe that the distributional gradient $G\colon \mathrm{L}^2(\Omega) \to \mathrm{H}^{-1}_{\partial\Omega}(\Omega)^d$ is a continuous linear operator, as for all $\varphi \in \mathrm{L}^2(\Omega)$ we have (recall Proposition 1)

$$\big|\langle G\varphi, \xi\rangle\big| := \left| -\int_\Omega \varphi \operatorname{div} \xi \,\mathrm{dx}\right| \le C\|\varphi\|_{\mathrm{L}^2(\Omega)}\|\xi\|_{\mathrm{H}^1(\Omega)^d} \quad \text{for all } \xi \in \mathrm{C}^\infty_c(\Omega)^d.$$

Moreover, the distributional gradient G restricted to $\mathrm{H}^1(\Omega)$ agrees exactly with the weak gradient ∇ on $\mathrm{H}^1(\Omega)$ per partial integration and the fundamental lemma of the calculus of variations. Hence, we are able to interpolate the operator (which we agree to call ∇ from now on) which by Theorem 3 and Lemma 2 yields that

$$\nabla \in \mathcal{L}\Big(\big[\mathrm{L}^2(\Omega), \mathrm{H}^1_F(\Omega)\big]_{1-\sigma}; \big[\mathrm{H}^{-1}_{\partial\Omega}(\Omega)^d, \mathrm{L}^2(\Omega)^d\big]_{1-\sigma}\Big) = \mathcal{L}\big(\mathrm{H}^{1-\sigma}_F(\Omega); \mathrm{H}^{-\sigma}(\Omega)^d\big).$$

Here, we have used coordinate-wise interpolation in the second component (cf. [6, Cor. 1.3.8]) and the fundamental interpolation property $[X_0, X_1]_\theta = [X_1, X_0]_{1-\theta}$ for any interpolation couple (X_0, X_1) and $0 < \theta < 1$, see [25, Thm. 1.9.3 b)].

We finally prove the main theorem.

Proof (Theorem 1) We had already noted below Proposition 1 that the forms

$$\mathrm{H}^1_{D_j}(\Omega) \times \mathrm{H}^1_{D_i}(\Omega) \ni (\varphi, \xi) \mapsto \big\langle -\nabla \cdot A^{i,j}\nabla\varphi, \xi\big\rangle := \big(A^{i,j}\nabla\varphi, \nabla\xi\big)_{\mathrm{L}^2(\Omega)}$$

are continuous for $i, j \in \{1, \dots, n\}$. We extend them to $\mathrm{H}^{\varepsilon+1}_{D_j}(\Omega) \times \mathrm{H}^{1-\varepsilon}_{D_i}(\Omega)$ using Lemma 3 as follows, thereby also extending $-\nabla \cdot A\nabla$ to a continuous operator from $\mathbb{H}^{\varepsilon+1}_D(\Omega)$ to $\mathbb{H}^{\varepsilon-1}_D(\Omega)$, cf. (2):

Let $i, j \in \{1, \dots, n\}$ be given and denote by $M_{i,j}$ the norm of $A^{i,j}$ when the latter is considered as a multiplier acting on $\mathrm{H}^\varepsilon(\Omega)^d$. Since $\mathrm{H}^\varepsilon(\Omega)^d$ is dense in $\mathrm{L}^2(\Omega)^d$, we estimate

$$\begin{aligned}\big|\big\langle -\nabla \cdot A^{i,j}\nabla\varphi, \xi\big\rangle\big| &= \big|\big(A^{i,j}\nabla\varphi, \nabla\xi\big)_{\mathrm{L}^2(\Omega)^d}\big| \le \|A^{i,j}\nabla\varphi\|_{\mathrm{H}^\varepsilon(\Omega)^d}\|\nabla\xi\|_{\mathrm{H}^{-\varepsilon}(\Omega)^d} \\ &\le M_{i,j}\|\nabla\varphi\|_{\mathrm{H}^\varepsilon(\Omega)^d}\|\nabla\xi\|_{\mathrm{H}^{-\varepsilon}(\Omega)^d} \le CM_{i,j}\|\varphi\|_{\mathrm{H}^{\varepsilon+1}_{D_j}(\Omega)}\|\xi\|_{\mathrm{H}^{1-\varepsilon}_{D_i}(\Omega)}\end{aligned}$$

for all $\varphi \in \mathrm{H}^{\varepsilon+1}_{D_j}(\Omega)$ and $\xi \in \mathrm{H}^1_{D_i}(\Omega)$ using Lemma 3. As $\mathrm{H}^1_{D_i}(\Omega)$ is again dense in $\mathrm{H}^{1-\varepsilon}_{D_i}(\Omega)$, we obtain a unique continuous linear extension of $-\nabla \cdot A^{i,j}\nabla$ to a mapping from $\mathrm{H}^{\varepsilon+1}_{D_j}(\Omega)$ to $\mathrm{H}^{\varepsilon-1}_{D_i}(\Omega)$. By definition (see (2)), this also gives a unique continuous linear extension of $-\nabla \cdot A\nabla$ to a mapping from $\mathbb{H}^{\varepsilon+1}_D(\Omega)$ to $\mathbb{H}^{\varepsilon-1}_D(\Omega)$.

For the extension of $-\nabla \cdot A\nabla$ to an operator $\mathbb{H}^{1-\varepsilon}_D(\Omega) \to \mathbb{H}^{-1-\varepsilon}_D(\Omega)$, we observe that taking the conjugate transpose $(A^{i,j})^*$ of $A^{i,j}$ preserves the multiplier property for $\mathrm{H}^\varepsilon(\Omega)^d$. So analogously to the above we obtain

$$\big|\big\langle -\nabla \cdot A^{i,j}\nabla\varphi, \xi\big\rangle\big| = \big|\big(\nabla\varphi, (A^{i,j})^*\nabla\xi\big)_{\mathrm{L}^2(\Omega)^d}\big| \le CM^*_{i,j}\|\varphi\|_{\mathrm{H}^{1-\varepsilon}_{D_j}(\Omega)}\|\xi\|_{\mathrm{H}^{1+\varepsilon}_{D_i}(\Omega)}$$

for $\varphi \in \mathrm{H}^1_{D_j}(\Omega)$ and $\xi \in \mathrm{H}^{1+\varepsilon}_{D_i}(\Omega)$, where $M^*_{i,j}$ denotes the multiplier norm of $(A^{i,j})^*$. This implies that $-\nabla \cdot A\nabla$ extends to a continuous linear operator from $\mathbb{H}^{1-\varepsilon}_D(\Omega)$ to $\mathbb{H}^{-1-\varepsilon}_D(\Omega)$. Hence the operator is compatible with the interpolation couples $(\mathbb{H}^{1+\varepsilon}_D(\Omega), \mathbb{H}^{1-\varepsilon}_D(\Omega))$ and $(\mathbb{H}^{\varepsilon-1}_D(\Omega), \mathbb{H}^{-1-\varepsilon}_D(\Omega))$, and this is then clearly also true for $-\nabla \cdot A\nabla + \gamma$ for any $\gamma \geq 0$.

Now observe that $-\nabla \cdot A\nabla + \gamma \in \mathcal{L}_{\mathrm{iso}}(\mathbb{H}^1_D(\Omega); \mathbb{H}^{-1}_D(\Omega))$ for $\gamma > \mu$ by the Gårding inequality assumption and the Lax-Milgram lemma, and that

$$\big[\mathbb{H}^{1+\varepsilon}_D(\Omega), \mathbb{H}^{1-\varepsilon}_D(\Omega)\big]_{\frac{1}{2}} = \mathbb{H}^1_D(\Omega) \quad \text{and} \quad \big[\mathbb{H}^{\varepsilon-1}_D(\Omega), \mathbb{H}^{-1-\varepsilon}_D(\Omega)\big]_{\frac{1}{2}} = \mathbb{H}^{-1}_D(\Omega)$$

due to Theorem 3 and Lemma 2 (and again coordinate-wise interpolation, see [6, Cor. 1.3.8]). But then the stability result of Šneĭberg as in Theorem 2 tells us that there exists $0 < \delta \leq \varepsilon$ such that we still have $-\nabla \cdot A\nabla + \gamma \in \mathcal{L}_{\mathrm{iso}}(\mathbb{H}^{\theta+1}_D(\Omega); \mathbb{H}^{\theta-1}_D(\Omega))$ for all $|\theta| < \delta$. This was the first claim. Note that if $D \neq \emptyset$, then $\gamma = \mu$ is also allowed due to the Poincaré inequality, cf. Remark 1.

For the claimed uniformity of δ and the norm of the inverse of $-\nabla \cdot A\nabla + \gamma$, we refer to [6, Ch. 1.3.5] and Remark 6.

Proof (Corollary 1) It is a mere reformulation of assertion (3) in Theorem 1 that for every $f \in \mathbb{H}^{\theta-1}_D(\Omega)$ there exists a unique $u \in \mathbb{H}^{\theta+1}_D(\Omega)$ satisfying the elliptic system equation (4) with $\|u\|_{\mathbb{H}^{\theta+1}_D(\Omega)} \leq C\|f\|_{\mathbb{H}^{\theta-1}_D(\Omega)}$, where C is independent of f.

Now let $\eta \geq 0$ and $p \geq 2$ be such that $\theta \geq \eta + d(\frac{1}{2} - \frac{1}{p})$, and consider $j \in \{1, \dots, n\}$. Then, for every function $U_j \in \mathrm{H}^{\theta+1}_{D_j}(\mathbb{R}^d)$ with the property that $(U_j)_{\restriction \Omega} = u_j$ we use the well known (generalized) Sobolev embeddings (cf. [25, Ch. 2.8.1]) as follows:

$$\|u_j\|_{\mathrm{H}^{1+\eta,p}_{D_j}(\Omega)} \leq \|U_j\|_{\mathrm{H}^{1+\eta,p}_{D_j}(\mathbb{R}^d)} \leq C^\star \|U_j\|_{\mathrm{H}^{\theta+1}_{D_j}(\mathbb{R}^d)}.$$

But this implies that $\|u_j\|_{\mathrm{H}^{1+\eta,p}_{D_j}(\Omega)} \leq C^\star \|u_j\|_{\mathrm{H}^{\theta+1}_{D_j}(\Omega)}$ and of course accordingly $\|u\|_{\mathbb{H}^{1+\eta,p}_D(\Omega)} \leq C^\star \|u\|_{\mathbb{H}^{\theta+1}_D(\Omega)}$, so the claim follows by observing that if we choose $0 < \eta < \theta$, then we are also allowed to choose $p > 2$ while still obeying the inequality $\theta \geq \eta + d(\frac{1}{2} - \frac{1}{p})$. Uniformity of the collected constants as claimed in the statement of the corollary finally follows immediately from the corresponding assertion in Theorem 1.

7 Application: a fracture model

As an application, we consider a standard phase-field model for brittle fracture as given in [4]. For the following exposition, we consider the formulation given in [20], where the fracture irreversibility is relaxed by a penalty approach. After introduction of a time-discretization, the evolution is given by a sequence of problems associated to each time-step. Namely, for a bounded domain $\Omega \subset \mathbb{R}^2$

satisfying Assumption 2, one searches for a (vector-valued) displacement $u \in \mathbb{H}^1_D(\Omega)$ and a (scalar) phase-field $\phi \in \mathrm{H}^1(\Omega)$ solving the system of equations

$$\left.\begin{aligned} \big(g(\phi)e(u):e(v)\big) &= \ell(v), \\ \big(\epsilon^{-1}(\phi-1)+(1-\kappa)\big(\phi e(u):e(u)\big)+\varrho[(\phi-\phi^-)^+]^3,\psi\big)_{\mathrm{L}^2(\Omega)} & \\ +\big\langle -\nabla\cdot\epsilon\nabla\phi,\psi\big\rangle &= 0 \end{aligned}\right\} \tag{7}$$

for all $v \in \mathbb{H}^1_D(\Omega)$ and $\psi \in \mathrm{H}^1(\Omega)$, with given loads $\ell \in \mathbb{H}^{\theta_0-1}_D(\Omega)$ for some $\theta_0 > 0$, ϕ^- satisfying $0 \le \phi^- \le 1$, with $0 < \kappa \ll \epsilon \ll 1$ and $\varrho > 0$ as well as $g(\phi) = (1-\kappa)\phi^2 + \kappa$, where $e(u)$ and $e(v)$ denotes the symmetric gradient of u and v, respectively. It has been shown in [20] that this problem admits a Hilbert space solution $(u,\phi) \in \mathbb{H}^1_D(\Omega) \times \mathrm{H}^1(\Omega)$ with the additional regularity $u \in \mathbb{W}^{1,p}(\Omega)$ for some $p > 2$ and $\phi \in \mathrm{L}^\infty(\Omega)$; in fact, $0 \le \phi(\mathrm{x}) \le 1$ holds for almost all $\mathrm{x} \in \Omega$.

With the results obtained in this work, we can now show the following improved differentiability result.

Corollary 2 *There exists* $0 < \bar\theta \le \theta_0$ *such that the solution* $(u,\phi) \in \big(\mathbb{W}^{1,p}(\Omega) \cap \mathbb{H}^1_D(\Omega)\big) \times \big(\mathrm{H}^1(\Omega) \cap \mathrm{L}^\infty(\Omega)\big)$ *of* (7) *admits the additional regularity* $u \in \mathbb{H}^{\theta+1}_D(\Omega)$ *and* $\phi \in \mathrm{H}^{\theta+1}(\Omega)$ *for any* θ *satisfying* $0 < \theta \le \bar\theta$. *Moreover we obtain the estimate*

$$\|u\|_{\mathbb{H}^{1+\theta}_D(\Omega)} \le C\|\ell\|_{\mathbb{H}^{\theta_0-1}_D(\Omega)}$$

with a constant $C = C(\|\ell\|^2_{\mathbb{H}^{-1,p}_D(\Omega)}, \varrho, \epsilon)$.

Proof Slightly rewriting the second equation in (7), we see that ϕ satisfies

$$\big(-\nabla\cdot\epsilon\nabla+\epsilon^{-1}\big)\phi = \epsilon^{-1} + (\kappa-1)\big(\phi e(u):e(u)\big) - \varrho[(\phi-\phi^-)^+]^3 \quad \text{in } \mathrm{H}^{-1}(\Omega).$$

By the regularity $\phi \in \mathrm{L}^\infty(\Omega)$ and $u \in \mathbb{W}^{1,p}(\Omega)$ it is clear that the right hand side is in fact an element of $\mathrm{L}^{p/2}(\Omega)$. Consequently, by Sobolev embedding, there exists some $\vartheta > 0$ such that it is an element of $\mathrm{H}^{\vartheta-1}(\Omega)$. Theorem 1 then shows that we have $\phi \in \mathrm{H}^{\theta+1}(\Omega)$ for all $0 < \theta \le \bar\vartheta$ for some $\bar\vartheta \le \vartheta$, and standard Sobolev embedding theorems assert that $\phi \in \mathrm{C}^\sigma(\Omega)$ for $\sigma = 1+\theta-\frac{2}{p}$. Moreover, by [20, Corollary 4.2], we have that $\|\phi e(u):e(u)\|_{\mathrm{L}^{p/2}(\Omega)} \le c\|\ell\|^2_{\mathbb{H}^{-1,p}_D(\Omega)}$ for some constant $c \ge 0$, and thus

$$\|\phi\|_{\mathrm{C}^\sigma(\Omega)} \le c\big(\|\ell\|^2_{\mathbb{H}^{-1,p}_D(\Omega)} + \varrho + \epsilon^{-1}\big).$$

But then, by definition, $g(\phi) \in \mathrm{C}^\sigma(\Omega)$ too and Lemma 1 (iii) shows that this is indeed a multiplier on $\mathbb{H}^\theta(\Omega)$. Now another application of Theorem 1 to the equation

$$\big(g(\phi)e(u):e(v)\big) = \ell(v) \quad \text{for all } v \in \mathbb{H}^1_D(\Omega)$$

yields the claimed regularity. For the stability estimate, we utilize the above bound on $\|\phi\|_{\mathrm{C}^\sigma(\Omega)}$ together with the uniformity assertion in Corollary 1.

Such regularity results can now be utilized, e.g., in the numerical analysis of such fracture processes, as the improved regularity $\mathbb{H}_D^{1+\theta}(\Omega)$ allows to quantify best-approximation errors in $\mathbb{H}_D^1(\Omega)$ in terms of the discretization fineness.

Remark 7 In the case where the irreversibility of the fracture is not relaxed via a penalization approach, the equation for ϕ becomes an obstacle problem where the term involving $\varrho([(\phi-\phi^-)^+]^3$ is replaced by the requirement $\phi \le \phi^-$. If the domain is sufficiently regular, then classical $\mathrm{W}^{2,p/2}(\Omega)$-regularity of the obstacle problem, i.e., $\phi \in \mathrm{W}^{2,p/2}(\Omega)$ as long as $\phi^- \in \mathrm{W}^{2,p/2}(\Omega)$, can be used to show that ϕ is again a multiplier (see, e.g., [5, Corollary II.3]).

8 Application: quasilinear equations

We give another possible application for Theorem 1. Let us consider a single abstract parabolic quasilinear evolution equation of the form

$$u'(t) - \nabla \cdot A(u(t))\nabla u(t) = F(u(t)), \quad u(0) = u_0. \tag{8}$$

For this exposition, we assume that the (nonlinear) functions A and F and the initial value u_0 are suitably regular. Suppose we want to treat (8) in the space $\mathrm{H}_D^{\theta-1}(\Omega)$ for $\Omega \subset \mathbb{R}^2$. (We give a motivation why this is interesting below Remark 8.)

A possible way to do so are the abstract frameworks of Amann [2] and Prüss [21, Thm. 3.1] basing on maximal regularity techniques. One of the most critical points to verify for this is that there is a space $\mathcal{D}_\theta$ such that $-\nabla \cdot A(u)\nabla$ is a continuous linear operator from $\mathcal{D}_\theta$ to $\mathrm{H}_D^{\theta-1}(\Omega)$ for *all* u from the trace- or interpolation space $(\mathcal{D}_\theta, \mathrm{H}_D^{\theta-1}(\Omega))_{1/p,p}$ for some $1 < p < \infty$. Here our main Theorem 1 comes into play: If we are able to show that $u \in (\mathrm{H}_D^{1+\theta}(\Omega), \mathrm{H}_D^{\theta-1}(\Omega))_{1/p,p}$ implies that $A(u)$ is a suitable multiplier on $\mathrm{H}^\varepsilon(\Omega)^2$ for some $\theta < \varepsilon < \frac{1}{2}$, then the theorem shows that $\mathcal{D}_\theta = \mathrm{H}_D^{1+\theta}(\Omega)$ indeed does the job.

Indeed, assuming that the set $\Omega \cup D \subset \mathbb{R}^2$ is regular in the sense of Gröger (cf. Remark 3) and using the results in [10, Sect. 3], one may show by interpolation techniques that

$$\big(\mathrm{H}_D^{1+\theta}(\Omega), \mathrm{H}_D^{\theta-1}(\Omega)\big)_{\frac{1}{p},p} \hookrightarrow \mathrm{B}^\varepsilon_{\frac{d}{\varepsilon}(1+\eta),2}(\Omega)$$

if

$$\theta < \varepsilon < \frac{1+\eta}{\eta}\theta \quad \text{and} \quad p > \frac{2(1+\theta)}{1+\theta-\varepsilon\frac{\eta}{1+\eta}-\frac{d}{2}}$$

with η sufficiently large, but finite. If we then further suppose that A is sufficiently regular to pass the regularity of u to $A(u)$, then Theorem 1 and Lemma 1 tell us that $-\nabla \cdot A(u)\nabla$ is indeed a continuous linear operator from $\mathrm{H}_D^{1+\theta}(\Omega)$ to $\mathrm{H}_D^{\theta-1}(\Omega)$ for every $u \in (\mathrm{H}_D^{1+\theta}(\Omega), \mathrm{H}_D^{\theta-1}(\Omega))_{1/p,p}$ for p as given above. The condition on p shows why we needed to restrict ourselves to $d = 2$ here, as $1 - \frac{d}{2} + \theta < 0$ for $d \ge 3$ due to $\theta < \frac{1}{2}$.

So, while there are still quite formidable assumptions left to establish for the frameworks in [2] and [21] to completely treat the quasilinear equation in $\mathrm{H}_D^{\theta-1}(\Omega)$, Theorem 1 can be used as a starting point to do so as explained above.

Remark 8 Let us point out that the space $\mathrm{B}^{\varepsilon}_{\infty,2}(\Omega)$, so $\eta = \infty$ in $B^{\varepsilon}_{d(1+\eta)/\varepsilon,2}(\Omega)$, is not suitable as a multiplier space here since $\mathrm{H}_D^{1+\theta}(\Omega) \hookrightarrow \mathrm{B}^{\theta}_{\infty,2}(\Omega)$ and the smoothness order in the Besov space cannot be improved ([23, Thm. 2.2.3]). This is already insufficient for the multiplier property on $\mathrm{H}^{\varepsilon}(\Omega)$ due to the requirement $\varepsilon > \theta$ and will not improve by interpolation with $\mathrm{H}_D^{\theta-1}(\Omega)$. In this sense, we really need $\mathrm{B}^{\varepsilon}_{d(1+\eta)/\varepsilon,2}(\Omega)$ with finite η as a multiplier space for $\mathrm{H}^{\varepsilon}(\Omega)$ here.

Finally, let us briefly explain why it is of interest to treat the quasilinear equation in $\mathrm{H}_D^{1+\theta}(\Omega)$ in space dimension $d = 2$: Consider a bounded function $u\colon [0,T] \to (\mathrm{H}_D^{1+\theta}(\Omega), \mathrm{H}_D^{\theta-1}(\Omega))_{1/p,p}$. For p suitably large, we expect that $|\nabla u(t)|^2$ admits enough integrability to give rise to an element of $\mathrm{H}_D^{\theta-1}(\Omega)$, since then the interpolation space is sufficiently close to $\mathrm{H}_D^{1+\theta}(\Omega)$ to still embed into a space of type $\mathrm{H}_D^{1,p}(\Omega)$ with $p > d = 2$, as in the proof of Corollary 1. Hence, treating the quasilinear equation in $\mathrm{H}_D^{\theta-1}(\Omega)$ in space dimension $d = 2$ would allow to incorporate quadratic gradient terms of $u(t)$ in $F(u(t))$ which is of practical relevance. Here, we have assumed u to be bounded so that there occurs no loss of time integrability over the finite time interval $[0,T]$ between $u(\cdot)$ in the interpolation space and $|\nabla u(\cdot)|^2$ in $\mathrm{H}_D^{\theta-1}(\Omega)$. To achieve this boundedness in the maximal regularity setting we need to consider u with values in the interpolation space.

Acknowledgements The authors express their gratitude to Joachim Rehberg (WIAS Berlin) for valuable discussions. Hannes Meinlschmidt is grateful for support of his former institution TU Darmstadt; Winnifried Wollner acknowledges funding by the DFG priority program 1962.

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