
Exercise 1 (Existence of globally optimal solutions)**.** Determine whether the following optimization problems in function spaces admit a globally optimal solution.

$$\min_{u \in C([0,1])} \int_0^1 u(x)^2 \, \mathrm{d}x \quad \text{s.t.} \quad u(1) = 1, \tag{P1}$$

where $C([0,1])$ is the Banach space of all continuous functions $u\colon [0,1] \to \mathbb{R}$ equipped with the norm $\|u\|_\infty := \max_{x \in [0,1]} |u(x)|$,

$$\min_{u \in L^2(0,1)} -\int_0^1 x\, u(x)^2 \, \mathrm{d}x \quad \text{s.t.} \quad \|u\|_{L^2(0,1)} \leq 1, \tag{P2}$$

and

$$\max_{y \in H^1(0,1)} \|y\|_{L^\infty(0,1)} \quad \text{s.t.} \quad \|y\|_{H^1(0,1)} \leq 2, \tag{P3}$$

where $H^1(0,1)$ is the Sobolev (Hilbert) space $H^1(0,1) := \{y \in L^2(0,1)\colon y' \in L^2(0,1)\}$ equipped with the norm $\|y\|_{H^1(0,1)} := \|y\|_{L^2(0,1)} + \|y'\|_{L^2(0,1)}$.

Hint: The natural embedding $W^{1,2}(0,1) = H^1(0,1) \hookrightarrow C([0,1])$ induced by the identity mapping $u \mapsto u$ is a *compact* linear operator, see Exercise 4 below.

Solution. Problem (P1) does not admit a globally optimal solution. Denote the objective function on $C([0,1])$ by f and the feasible set by $\mathcal{F} := \{u \in C([0,1])\colon u(1) = 1\}$. It is clear that 0 is a lower bound for f and the sequence $u_k(x) := x^k \in \mathcal{F}$ satisfies $f(u_k) = \frac{1}{2k+1} \to 0$ as k goes to infinity. Hence $\inf_{u \in \mathcal{F}} f(u) = 0$. But there is no function $\bar{u}$ which satisfies $f(\bar{u}) = 0$, because for every function $u \in \mathcal{F}$, there exists $\delta > 0$ sufficiently small such that $u(x) \geq \frac{1}{2}$ for all $x \in [1-\delta, 1]$ due to continuity of u und $u(1) = 1$. This implies that $f(u) \geq \frac{\delta}{4} > 0$ for every feasible $u \in \mathcal{F}$.

Problem (P2) also admits no globally optimal solution. Let again f be the objective function, this time on $L^2(0,1)$, and let $\mathcal{F} := \{u \in L^2(0,1)\colon \|u\|_{L^2} \leq 1\}$ be the feasible set. Due to $0 \leq xu(x)^2 \leq u(x)^2$ almost everywhere in $(0,1)$, we have $f(u) \geq -\|u\|_{L^2(0,1)} \geq -1$ for every feasible function $u \in \mathcal{F}$. Moreover, -1 is indeed the infimum of f over $\mathcal{F}$, as the sequence $u_k(x) = \sqrt{k}\chi_{(1-\frac{1}{k},1)} \in \mathcal{F}$ demonstrates. Again, there is no feasible function attaining the minimum: The zero function is immediately discarded due to $f(0) = 0$, and for every nonzero $u \in \mathcal{F}$, we have $0 < xu(x)^2 < u(x)^2$ for all x from the non-null set $\{x\colon u(x) \neq 0\}$. But this means $f(u) > -\|u\|_{L^2(0,1)} \geq -1$ and the minimum cannot be attained.

Finally, problem (P3) admits a globally optimal solution. The Hilbert space $H^1(0,1)$ is certainly reflexive and the feasible set $\mathcal{F}$ is bounded, closed and convex, hence weakly compact in that space. Moreover, due to the continuity of the embedding $H^1(0,1) \hookrightarrow L^\infty(0,1)$, we know that there exists a number $C > 0$ such that $\|y\|_{L^\infty(0,1)} \le C\|y\|_{H^1(0,1)}$ for every function $y \in H^1(0,1)$, such that the objective function $f(y) = \|y\|_{L^\infty(0,1)}$ is bounded by $2C$ over $\mathcal{F}$. Accordingly, there exists a maximizing sequence $(y_k) \subset \mathcal{F}$ such that $f(y_k) \to f^* = \inf_{y\in\mathcal{F}} f(y) \le 2C < \infty$. Since $\mathcal{F}$ was weakly compact in $H^1(0,1)$, there exists a weakly convergent subsequence (y_{k_ℓ}) with some limit $\bar{y} \in \mathcal{F}$. Applying Lemma 2.6 from the lecture notes to the compact embedding $H^1(0,1) \hookrightarrow L^\infty(0,1)$ shows that (y_{k_ℓ}) converges in norm in $L^\infty(0,1)$. But this means by definition that $f(y_{k_\ell}) \to f(\bar{y})$ from which by uniqueness of limits it follows that $f(\bar{y}) = f^*$. Hence $\bar{y}$ is the global solution of (P3).

Exercise 2 (Continuity of superposition operators in Lebesgue-spaces)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real function and let X be a function space consisting of real-valued functions defined on a bounded open set $\Omega \subseteq \mathbb{R}^n$. Then the *superposition* or *Nemytskii* operator F (on X) induced by f is given by the mapping $X \ni u \mapsto f \circ u$, i.e., $F(u)(x) := f(u(x))$ as a function of $x \in \Omega$.

(a) Let $1 \le p, q < \infty$ and assume that f is continuous and satisfies

$$\big|f(t)\big| \le C\big(|t|^{\frac{p}{q}} + 1\big) \tag{1}$$

for some constant $C \ge 0$. Show that F is a sequentially continuous mapping from $L^p(\Omega)$ to $L^q(\Omega)$.

Hint: From the proof of the Riesz-Fischer theorem (completeness of L^p): Every L^p convergent sequence admits a subsequence which converges in a pointwise sense almost everywhere and which is uniformly bounded by an L^p function.

(b) Let $\Omega = (0,1)$ and assume that F is weakly sequentially continuous from $L^p(\Omega)$ to $L^q(\Omega)$, i.e., if $u_k \rightharpoonup u$ in $L^p(\Omega)$, then $F(u_k) \rightharpoonup F(u)$ in $L^q(\Omega)$. Show that f must already be an *affine-linear* function.

Hint: Use Rademacher's functions from Exercise 3.

(c) Let $1 < p < \infty$, let Ω be bounded with a Lipschitz boundary, and assume that f is Lipschitz-continuous (in particular, f satisfies (1) for $q = p$). Show that F is weakly sequentially continuous as a mapping from $W^{1,p}(\Omega)$ to itself. Discuss the difference to the previous case.

Hint: The properties of Ω imply the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ (this is the Rellich-Kondrachov theorem).

Solution.

(a) Let $(u_k) \subset L^p(\Omega)$ be a convergent sequence with limit $u \in L^p(\Omega)$. From the growth bound on f as in (1), we know that $F(u_k) \in L^q(\Omega)$, and by the hint, there exists a subsequence (u_{k_ℓ}) such that $u_{k_\ell}(x) \to u(x)$ for almost every $x \in \Omega$. But then the dominated convergence theorem implies that $F(u_{k_\ell})$ converges to $F(u)$ in $L^q(\Omega)$, and the assumptions of that theorem are satisfied since f is continuous, so $F(u_{k_\ell})$ also converges in a pointwise sense almost everywhere in Ω, and we obtain an $L^q(\Omega)$-bound for the sequence $F(u_{k_\ell})$ again by (1).

Since we can replace the original sequence (u_k) by any of its subsequences and obtain the same conclusion, we find that indeed $F(u_k)$ in total converges to $F(u)$ in $L^q(\Omega)$ by the *nitpicker lemma*: A sequence (a_k) converges to the limit a if and only if every subsequence of (a_k) admits a subsequence which converges to a (work this out!).

(b) We take the Rademacher function

$$u(x) := \begin{cases} \alpha & \text{if } x \in (0, \frac{1}{2}), \\ \beta & \text{if } x \in (\frac{1}{2}, 1). \end{cases}$$

Then, as in Exercise 3, $u_k \rightharpoonup \frac{1}{2}(\alpha + \beta)$ with $u_k(x) := u(kx)$ for $x \in (0,1)$. On the other hand, $F(u)$ is again a Rademacher function and $(F(u))_k = F(u_k)$, hence also $F(u_k) \rightharpoonup \frac{1}{2}(F(\alpha) + F(\beta))$. But then the assumption on weak continuity of F implies that

$$F\big(\tfrac{1}{2}(\alpha + \beta)\big) = \frac{1}{2}\big(F(\alpha) + F(\beta)\big),$$

and this means exactly that f is an affine function, since the preceding argument works for any $\alpha, \beta \in \mathbb{R}$.

(c) We have already seen in the part (a) of this exercise that F maps $L^p(\Omega)$ into itself. Moreover, the Lipschitz property of f implies that $\nabla F(u) = f'(u)\nabla u \in L^p(\Omega)^n$ if $u \in W^{1,p}(\Omega)$, hence F indeed maps $W^{1,p}(\Omega)$ into itself.

Now let $u_k \rightharpoonup u$ in $W^{1,p}(\Omega)$. The hint implies that $u_k \to u$ in $L^p(\Omega)$ (Lemma 2.6 in the lecture notes) and thus $F(u_k) \to F(u)$ in $L^p(\Omega)$ by part (a) of this exercise. On the other hand, $\nabla F(u_k) = f'(u_k)\nabla u_k$ is also bounded in $L^p(\Omega)^n$ by boundedness of the weakly convergent sequence (u_k) in $W^{1,p}(\Omega)$, so $(F(u_k))$ is indeed a bounded sequence in $W^{1,p}(\Omega)$. But then reflexivity of $W^{1,p}(\Omega)$ implies that there exists a weakly convergent subsequence $F(u_{k_\ell}) \rightharpoonup v \in W^{1,p}(\Omega)$. Using the hint again, we find $v = F(u)$, and again a subsequence-subsequence argument as in part (a) of this exercise shows that indeed the whole sequence $(F(u_k))$ converges weakly to $F(u)$.

Exercise 3 (An interesting family of functions (Rademacher))**.** Let $1 < p < \infty$ and let $f \in L^p_{\text{loc}}(\mathbb{R})$, that is, $f \in L^p(K)$ for every compact set $K \Subset \mathbb{R}$. Assume that $f(x+T) =$

$f(x)$ for almost every $x \in \mathbb{R}$, so f is T-periodic with $T > 0$. Set

$$\overline{f} := T^{-1} \int_0^T f(x)\, \mathrm{d}x$$

and consider the sequence $(u_k) \subset L^p(0,1)$ defined by

$$u_k(x) := f(kx), \quad x \in (0,1).$$

(a) Show that $u_k \rightharpoonup \overline{f}$ in $L^p(0,1)$.

Hint: It is sufficient to show the assertion for dual pairs with step functions in $L^{p'}(0,1)$ (why?).

(b) Examine the following examples:

(i) $f(x) = \sin(x)$,

(ii) f is 1-periodic given by

$$f(x) := \begin{cases} \alpha & \text{if } x \in (0, \frac{1}{2}), \\ \beta & \text{if } x \in (\frac{1}{2}, 1) \end{cases}$$

for $\alpha, \beta \in \mathbb{R}$. Such functions are called *Rademacher's functions*.

Solution.

(a) Following the hint, we only need to show that

$$\int_a^b f(kx)\, \mathrm{d}x \quad \longrightarrow \quad (b-a)\overline{f}$$

for all $a, b \in [0,1]$, because this implies that $\langle u_k, \chi \rangle \to \langle \overline{f}, \chi \rangle$ for every step function χ and these are dense in $L^{p'}(0,1)$, such that indeed $\langle u_k, g \rangle \to \langle \overline{f}, g \rangle$ for all $g \in L^{p'}(0,1)$ follows. Integration by substitution shows that

$$\int_a^b f(kx)\, \mathrm{d}x = \frac{1}{k} \int_{ka}^{kb} f(x)\, \mathrm{d}x\,.$$

Now we nest the interval (ka, kb) within multiples of $(0,T)$-intervals to make use of the periodicity. Therefore we choose integers $\ell(k)$ and $m(k)$ such that

$$(\ell - 1)T \le ka \le \ell T \quad \text{and} \quad mT \le kb \le (m+1)T$$

and split the preceding integral:

$$\begin{aligned} \int_a^b f(kx)\, \mathrm{d}x &= \frac{1}{k} \left[\int_{ka}^{\ell T} f(x)\, \mathrm{d}x + \sum_{\ell \le i \le m-1} \int_{iT}^{(i+1)T} f(x)\, \mathrm{d}x + \int_{mT}^{kb} f(x)\, \mathrm{d}x \right] \\ &= \frac{m-\ell}{k} \int_0^T f(x)\, \mathrm{d}x + \frac{1}{k} \left[\int_{ka}^{\ell T} f(x)\, \mathrm{d}x + \int_{mT}^{kb} f(x)\, \mathrm{d}x \right]. \end{aligned}$$

Firstly, the residual integrals vanish as $k \to \infty$ due to

$$\frac{1}{k}\left|\int_{ka}^{\ell T} f(x)\;\mathrm{d}x + \int_{mT}^{kb} f(x).\;\mathrm{d}x\right| \le \frac{1}{k}\|f\|_{L^1(0,T)}.$$

For the "main" part, we need to show that $\frac{m-\ell}{k} \to \frac{b-a}{T}$ as $k \to \infty$. From the construction of $m = m(k)$ and $\ell = \ell(k)$, we find

$$\frac{m-\ell}{k} \le \frac{b-a}{T} \le \frac{m-\ell+2}{k} \quad \iff \quad 0 \le \frac{b-a}{T} - \frac{m-\ell}{k} \le \frac{2}{k} \quad \text{for every } k \in \mathbb{N},$$

and so indeed $\frac{m-\ell}{k} \to \frac{b-a}{T}$ as $k \to \infty$. Hence,

$$\int_a^b u_k(x)\;\mathrm{d}x = \int_a^b f(kx)\;\mathrm{d}x \quad \overset{k\to\infty}{\longrightarrow} \quad \frac{b-a}{T}\int_0^T f(x)\;\mathrm{d}x = (b-a)\overline{f}$$

as desired.

(b) (i) Here we have that $u_k(x) := \sin(kx)$ converges weakly to zero in every $L^p(0,1)$, since clearly $\sin \in L^\infty(\mathbb{R})$ and thus also $\sin \in L^p_{\mathrm{loc}}(\mathbb{R})$ for every $1 < p < \infty$. This is a particular example of a sequence whose pointwise limit is bogus but which converges weakly, even to zero.

(ii) For the Rademacher functions we find that $u_k \rightharpoonup \frac{1}{2}(\alpha + \beta)$, where $u_k(x) := f(kx)$, again for all $L^p(0,1)$ spaces for $1 < p < \infty$ due to $f \in L^\infty(\mathbb{R})$.

Exercise 4 (A particularly important compact embedding (Sobolev))**.** In Exercise 2, we have already used compactness of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. In fact, the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for such domains whenever $\frac{1}{n} + \frac{1}{q} > \frac{1}{p}$. For $p > n$, even more is true, which we establish exemplarily for $n = 1$ and $\Omega = (0,1)$.

(a) Show that, for every $1 \le p \le \infty$, the space $W^{1,p}(0,1)$ is a subset of $L^\infty(0,1)$ and that the embedding $W^{1,p}(0,1) \hookrightarrow L^\infty(0,1)$ is continuous, so

$$\|u\|_{L^\infty(0,1)} \le C\|u\|_{W^{1,p}(0,1)} = C\big(\|u\|_{L^p(0,1)} + \|u'\|_{L^p(0,1)}\big)$$

for some constant $C > 0$ independent of u.

Hint: The smooth functions $C^\infty([0,1])$ on $[0,1]$ are dense in $W^{1,p}(0,1)$.

(b) Refine the previous embedding by proving that for $p > 1$ we even have $W^{1,p}(0,1) \hookrightarrow C^{0,1-\frac{1}{p}}([0,1])$, where

$$C^{0,\alpha}([0,1]) := \Big\{u \in C([0,1])\colon \|u\|_{C^{0,\alpha}([0,1])} < \infty\Big\}$$

with

$$\|u\|_{C^{0,\alpha}([0,1])} := \|u\|_{C([0,1])} + \sup_{x\ne y\in[0,1]} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

is the α-*Hölder* space for $0 < \alpha \le 1$.

(c) Prove that every bounded sequence in $C^{0,\alpha}([0,1])$ admits a uniformly convergent subsequence, or equivalently, that the embedding $C^{0,\alpha}([0,1]) \hookrightarrow C([0,1])$ is compact.

 Hint: Recall the Arzelà-Ascoli theorem.

(d) Infer that for $p > 1$, the space $W^{1,p}(0,1)$ embeds *compactly* into every Hölder space $C^{0,\alpha}([0,1])$ for $0 < \alpha < 1 - \frac{1}{p}$ and into the space of uniformly continuous functions $C([0,1])$.

Solution.

(a) Let $u \in C^\infty([0,1])$. We use the mean value of integration to obtain a number $z \in [0,1]$ such that

$$u(z) = \int_0^1 u(x)\, \mathrm{d}x\,.$$

But then we have for every $y \in [0,1]$ using the fundamental theorem of calculus and Hölder's inequality:

$$\begin{aligned}|u(y)| \le |u(x) - u(z)| + |u(z)| &\le \int_0^1 |u'(x)|\, \mathrm{d}x + \int_0^1 |u(x)|\, \mathrm{d}x \\ &\le \|u'\|_{L^p(0,1)} + \|u\|_{L^p(0,1)} = \|u\|_{W^{1,p}(0,1)},\end{aligned}$$

hence $\|u\|_{L^\infty(0,1)} \le \|u\|_{W^{1,p}(0,1)}$ for all $u \in C^\infty([0,1])$. Since $C^\infty([0,1])$ is dense in $W^{1,p}(0,1)$, the inequality extends to all of $W^{1,p}(0,1)$ by continuity.

(b) Again via the fundamental theorem of calculus, we find for every $u \in W^{1,p}(0,1)$

$$|u(y) - u(z)| \le \int_z^y |u'(x)|\, \mathrm{d}x \le |y-z|^{1-\frac{1}{p}} \|u'\|_{L^p(0,1)},$$

which shows that u is continuous and that

$$\sup_{y \ne z \in [0,1]} \frac{|u(y) - u(z)|}{|y-z|^{1-\frac{1}{p}}} \le \|u'\|_{L^p(0,1)}.$$

Together with the embedding $W^{1,p}(0,1) \hookrightarrow L^\infty(0,1)$, this implies the assertion.

(c) We start with the hint: The Arzelà-Ascoli theorem says that a subset $\mathcal{F} \subset C([0,1])$ is relatively compact *if and only if* (!) it is bounded and uniformly equicontinuous, so there is a common modulus of continuity for all functions from $\mathcal{F}$. Rephrasing the latter in (ε, δ)-language, this means:

for every $\varepsilon > 0$ there exists $\delta > 0$:

$$\big(|x-y| < \delta \quad \Longrightarrow \quad |u(x) - u(y)| < \varepsilon \text{ for all } u \in \mathcal{F}\big).$$

Note that δ may depend on ε, but not on x, y or u.

Now, choosing a bounded sequence $(u_k) \subset C^{0,\alpha}([0,1])$ as the set $\mathcal{F} \subset C([0,1])$, it is clear from the definition of the Hölder norm $\|\cdot\|_{C^{0,\alpha}([0,1])}$ that $\mathcal{F}$ is bounded in $C([0,1])$. Moreover, again by boundedness in the Hölder norm, there is a number $C \geq 0$ such that

$$\sup_{x \neq y \in [0,1]} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq C \quad \text{for all } u \in \mathcal{F}.$$

But this implies exactly equicontinuity of functions in $\mathcal{F}$, with the choice $\delta := C^{-1}\sqrt[\alpha]{\varepsilon}$. Hence the Arzelà-Ascoli theorem implies that $\mathcal{F}$ is relatively compact in $C([0,1])$ and this means exactly that (u_k) admits a subsequence which converges in $C([0,1])$, i.e., uniformly.

(d) We have seen that $W^{1,p}(0,1) \hookrightarrow C^{0,1-\frac{1}{p}}([0,1])$ and the latter embeds *compactly* into $C([0,1])$ by the foregoing part of the exercise.

For the assertion within the Hölder scale, we show more generally that in fact, $C^{0,\beta}([0,1])$ embeds compactly $C^{0,\alpha}([0,1])$ whenever $0 < \alpha < \beta$: Let (u_k) be a bounded sequence in $C^{0,\beta}([0,1])$. Then there exists a uniformly convergent subsequence (u_{k_ℓ}). We show that (u_{k_ℓ}) even converges in $C^{0,\alpha}([0,1])$ by showing that it is a Cauchy sequence for the seminorm

$$[f]_\alpha := \sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

as follows:

$$\begin{aligned}
\left[u_{k_\ell} - u_{k_m}\right]_\alpha &\leq \left[u_{k_\ell} - u_{k_m}\right]_\beta^{\frac{\alpha}{\beta}} \cdot \left(2\left\|u_{k_\ell} - u_{k_m}\right\|_{C([0,1])}\right)^{1-\frac{\alpha}{\beta}} \\
&\leq (2M)^{\frac{\alpha}{\beta}} \left(2\left\|u_{k_\ell} - u_{k_m}\right\|_{C([0,1])}\right)^{1-\frac{\alpha}{\beta}},
\end{aligned}$$

where M is the bound on the sequence (u_k) in $C^{0,\beta}([0,1])$, and thus

$$\left[u_{k_\ell} - u_{k_m}\right]_\alpha \leq M^{\frac{\alpha}{\beta}} \left\|u_{k_\ell} - u_{k_m}\right\|_{C([0,1])}^{1-\frac{\alpha}{\beta}}.$$

Since (u_{k_ℓ}) was a convergent sequence in $C([0,1])$ it is in particular a Cauchy sequence there, hence so it is in $C^{0,\alpha}([0,1])$. But this was the claim.