Exercise 1 (Robinson's CQ for particular problems)**.** Let $\bar{x} \in \mathcal{F} = G^{-1}[K]$ be given with $G\colon X \to Z$ F-differentiable, where

$$G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} : X \to Z_1 \times Z_2 \quad \text{and} \quad K = K_1 \times K_2 \quad (K_i \subseteq Z_i,\ i = 1,2),$$

(a) Show that if $G'(\bar{x})$ is surjective, then (ACQ) is satisfied, i.e., surjectivity is a constraint qualification.

(b) Let $G_1'(\bar{x}) \in \mathcal{L}(X; Z_1)$ be surjective.

(i) Show that (RCQ) for $\bar{x}$ is equivalent to

$$0 \in \operatorname{int} \Big(G_2(\bar{x}) + G_2'(\bar{x})\big(G_1'(\bar{x})^{-1}\big[K_1 - G_1(\bar{x})\big]\big) - K_2 \Big). \tag{1}$$

(ii) Assume additionally that $\operatorname{int} K_2 \neq \emptyset$. Show that then (RCQ) for $\bar{x}$ is equivalent to the existence of $h \in X$ such that

$$\begin{aligned} G_1(\bar{x}) + G_1'(\bar{x})h &\in K_1, \\ G_2(\bar{x}) + G_2'(\bar{x})h &\in \operatorname{int} K_2. \end{aligned} \tag{2}$$

(c) Assume that $K_1 = \{0_{Z_1}\}$.

(i) Let the constraint $G_2(x) \in K_2$ be void, so non-existent or trivial. Show that then (RCQ) is satisfied in $\bar{x}$ if and only if $G_1'(\bar{x})$ is surjective, and that $T(\mathcal{F}, \bar{x}) = \ker G_1'(\bar{x})$ in this case.

Remark: This statement is also known as *Ljusternik's theorem*.

(ii) Let $\operatorname{int} K_2 \neq \emptyset$. Show that $\bar{x}$ satisfies (RCQ) if and only if $G_1'(\bar{x}) \in \mathcal{L}(X; Z_1)$ is surjective and there exists $h \in \ker G_1'(\bar{x})$ such that

$$G_2(\bar{x}) + G_2'(\bar{x})h \in \operatorname{int} K_2,$$

and that in this case

$$T(\mathcal{F}, \bar{x}) = \ker G_1'(\bar{x}) \cap T\big(G_2^{-1}[K_2], \bar{x}\big) = \ker G_1'(\bar{x}) \cap T_\ell(G_2, K_2, \bar{x}).$$

(iii) Give another proof of the equivalence of (RCQ) and the (MFCQ) for classical NLPs.

Exercise 2 (Polar cone)**.** Let $\emptyset \neq C \subseteq X$ be a given set and consider its polar cone

$$C^\circ := \{x' \in X^* : \langle x', x\rangle_{X^*,X} \leq 0 \text{ for all } x \in C\}.$$

(a) Show that C° is a nonempty closed convex cone.

(b) Show that $\overline{C}^\circ = C^\circ$.

(c) Now assume that C is convex. Show that $C^\circ = \operatorname{cone}(C)^\circ$.

Exercise 3 (Interior of an important cone)**.** Let X be a function space over the set $\Omega \subset \mathbb{R}^n$ and consider the cone of nonpositive functions in X:

$$K_- := \Big\{f \in X : f(x) \leq 0 \text{ for all } x \in \Omega\Big\}.$$

Determine whether K_- has nonempty interior for the choices $X = L^p(\Omega)$ for $1 \leq p \leq \infty$ and $X = C(\overline{\Omega})$.

Exercise 4 (Topological properties of convex sets)**.** Let $\emptyset \neq C \subseteq X$ be a convex set. For two points $x, y \in X$ we set

$$[x, y) = \{(1-\lambda)x + \lambda y : \lambda \in [0, 1)\},$$

so the set of convex combinations between x and y not including y.

(a) Show that $[x, y) \subset \operatorname{int} C$ for $x \in \operatorname{int} C$ and $y \in \overline{C}$. Infer that $\operatorname{int} C$ is convex.

Hint: It will be easier to first show the assertion for $y \in C$ and then extend the proof to $y \in \overline{C}$.

(b) Show that $\operatorname{int} C = \operatorname{int} \overline{C}$ if $\operatorname{int} C$ is nonempty.

Hint: There are multiple ways to solve this. Find at least two proofs. One possibility: Assume the contrary and use the geometric version of the Hahn-Banach theorem to construct a point $z \in \partial C$ with an open neighborhood whose intersection with C is empty (which is a contradiction, why?).

Remark: If the set C is even *convex-series* closed (*cs*-closed)—that means: For any sequences $(x_k) \subseteq C$ and $(\lambda_k) \geq 0$ with $\sum_{k=1}^\infty \lambda_k = 1$ for which $x = \sum_{k=1}^\infty \lambda_k x_k$ exists in X, we have $x \in C$—, then in fact $\operatorname{int} C = \operatorname{int} \overline{C}$. Such *cs*-closed sets are always trivially convex, and open or closed convex sets are also *cs*-closed. How does this fit with the assertion in (b)?