
Exercise 1 (Robinson's CQ for particular problems)**.** Let $\bar{x} \in \mathcal{F} = G^{-1}[K]$ be given with $G\colon X \to Z$ F-differentiable, where

$$G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} : X \to Z_1 \times Z_2 \quad \text{and} \quad K = K_1 \times K_2 \quad (K_i \subseteq Z_i,\ i = 1,2),$$

(a) Show that if $G'(\bar{x})$ is surjective, then (ACQ) is satisfied, i.e., surjectivity is a constraint qualification.

(b) Let $G_1'(\bar{x}) \in \mathcal{L}(X; Z_1)$ be surjective.

  (i) Show that (RCQ) for $\bar{x}$ is equivalent to

$$0 \in \operatorname{int}\Big(G_2(\bar{x}) + G_2'(\bar{x})\big(G_1'(\bar{x})^{-1}\big[K_1 - G_1(\bar{x})\big]\big) - K_2\Big). \tag{1}$$

  (ii) Assume additionally that $\operatorname{int} K_2 \neq \emptyset$. Show that then (RCQ) for $\bar{x}$ is equivalent to the existence of $h \in X$ such that

$$\begin{aligned} G_1(\bar{x}) + G_1'(\bar{x})h &\in K_1, \\ G_2(\bar{x}) + G_2'(\bar{x})h &\in \operatorname{int} K_2. \end{aligned} \tag{2}$$

(c) Assume that $K_1 = \{0_{Z_1}\}$.

  (i) Let the constraint $G_2(x) \in K_2$ be void, so non-existent or trivial. Show that then (RCQ) is satisfied in $\bar{x}$ if and only if $G_1'(\bar{x})$ is surjective, and that $T(\mathcal{F}, \bar{x}) = \ker G_1'(\bar{x})$ in this case.
  Remark: This statement is also known as *Ljusternik's theorem*.

  (ii) Let $\operatorname{int} K_2 \neq \emptyset$. Show that $\bar{x}$ satisfies (RCQ) if and only if $G_1'(\bar{x}) \in \mathcal{L}(X; Z_1)$ is surjective and there exists $h \in \ker G_1'(\bar{x})$ such that

$$G_2(\bar{x}) + G_2'(\bar{x})h \in \operatorname{int} K_2,$$

  and that in this case

$$T(\mathcal{F}, \bar{x}) = \ker G_1'(\bar{x}) \cap T\big(G_2^{-1}[K_2], \bar{x}\big) = \ker G_1'(\bar{x}) \cap T_\ell(G_2, K_2, \bar{x}).$$

  (iii) Give another proof of the equivalence of (RCQ) and the (MFCQ) for classical NLPs.

Solution. For convenience, we write down (RCQ) for $\bar{x} \in \mathcal{F} = G^{-1}[K]$ again:

$$0 \in \operatorname{int}\big(G(\bar{x}) + G'(\bar{x})X - K\big) = \operatorname{int}\left(\begin{pmatrix} G_1(\bar{x}) \\ G_2(\bar{x}) \end{pmatrix} + \begin{pmatrix} G_1'(\bar{x})X \\ G_2'(\bar{x})X \end{pmatrix} - \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}\right) \qquad \text{(RCQ)}$$

(a) If $G'(\bar{x})$ is surjective, then $0 \in \operatorname{int} G'(\bar{x})X = Z$. On the other hand, $Z = Z + G(\bar{x}) - K$ and thus

$$0 \in \operatorname{int}\big(G(\bar{x}) + G'(\bar{x})X - K\big),$$

so (RCQ) in $\bar{x}$ is satisfied. Since (RCQ) implies (ACQ), this shows the assertion.

(b) (i) (RCQ) $\implies$ (1): Let (RCQ) for $\bar{x}$ be satisfied. Then there exists $\varepsilon > 0$ such that for all $z = \binom{z_1}{z_2} \in Z_1 \times Z_2$ with $\|z_i\|_{Z_i} < \varepsilon$ there is $h \in X$ such that

$$z_i \in G_i(\bar{x}) + G_i'(\bar{x})h - K_i.$$

For $i = 1$, this means that $h \in G_1'(\bar{x})^{-1}\big[G_1(\bar{x}) - K_1 - z_1\big]$. Re-inserting shows that

$$z_2 \in G_2(\bar{x}) + G_2'(\bar{x})h - K_2 \subseteq G_2(\bar{x}) + G_2'(\bar{x})\Big(G_1'(\bar{x})^{-1}\big[G_1(\bar{x}) - K_1 - z_1\big]\Big) - K_2.$$

Now choosing $z_1 = 0$ and $z_2 \in B_{\varepsilon, Z_2}(0)$ arbitrarily yields

$$B_{\varepsilon, Z_2}(0) \subseteq G_2(\bar{x}) + G_2'(\bar{x})\Big(G_1'(\bar{x})^{-1}\big[G_1(\bar{x}) - K_1\big]\Big) - K_2,$$

which is exactly (1). We have not used surjectivity of $G_1'(\bar{x})$ for this implication.

For the reverse implication (1) $\implies$ (RCQ), let

$$0 \in \operatorname{int}\Big(G_2(\bar{x}) + G_2'(\bar{x})\Big(G_1'(\bar{x})^{-1}\big[G_1(\bar{x}) - K_1\big]\Big) - K_2\Big). \qquad (1)$$

Consider $z = \binom{z_1}{z_2} \in Z$. We need to show that there exists $h \in X$ such that

$$z_i \in G_i(\bar{x}) + G_i'(\bar{x})h - K_i$$

for $\|z_i\|_{Z_i}$ sufficiently small. Due to $G_1'(\bar{x})$ being surjective, there exists $h_1 \in X$ such that $G_1'(\bar{x})h_1 = z_1$ for every $z_1 \in Z_1$. Hence, we can restrict ourselves to searching $h_2 \in X$ such that

$$G_1'(\bar{x})h_2 \in K_1 - G_1(\bar{x}) \qquad \iff \qquad h_2 \in G_1'(\bar{x})^{-1}\big[K_1 - G_1(\bar{x})\big]$$

and

$$z_2 - G_2'(\bar{x})h_1 \in G_2(\bar{x}) + G_2'(\bar{x})h_2 - K_2$$

because then $h = h_1 + h_2$ is the searched-for element of X.

Assumption (1) indeed means exactly that there exists such an $h_2 \in X$, provided that the norm $\|z_2 - G_2'(\bar{x})h_1\|_{Z_2}$ is sufficiently small, say smaller than

$\varepsilon > 0$. Now, z_2 is not a problem because we are free to choose its norm as small as needed, for instance $\|z_2\|_{Z_2} \le \frac{\varepsilon}{4}$. For $\|G_2'(\bar{x})h_1\|_{Z_2}$, we argue why we can achieve

$$\|h_1\|_X \le \frac{\varepsilon}{4}\|G_2'(\bar{x})\|^{-1}_{\mathcal{L}(X;Z_2)}$$

such that, by continuity of $G_2'(\bar{x})$, we have

$$\|G_2'(\bar{x})h_1\|_{Z_2} \le \|G_2'(\bar{x})\|_{\mathcal{L}(X;Z_2)}\|h_1\|_X \le \frac{\varepsilon}{4}.$$

The open mapping theorem tells us that $0 \in \operatorname{int} G_1'(\bar{x})B_{r,X}(0)$ for any $r > 0$, so for every $r > 0$ there exists $\delta = \delta(r)$ such that $B_{\delta(r),Z}(0) \subset G_1'(\bar{x})B_{r,X}(0)$. Read upside down, this means that every z_1 of norm smaller than $\delta(r)$ can be expressed by $z_1 = G_1'(\bar{x})h_1$ with $\|h_1\|_X < r$. Now choosing

$$r = \frac{\varepsilon}{4}\|G_2'(\bar{x})\|^{-1}_{\mathcal{L}(X;Z_2)}$$

yields

$$\|z_2 - G_2'(\bar{x})h_1\|_{Z_2} \le \|z_2\|_{Z_2} + \|G_2'(\bar{x})h_1\|_{Z_2} \le \frac{\varepsilon}{2}$$

and thus

$$z_2 - G_2'(\bar{x})h_1 \in G_2(\bar{x}) + G_2'(\bar{x})h_2 - K_2.$$

(ii) The proof works exactly as the one of Lemma 3.17 from the lecture notes (equivalence of LSCQ and RCQ). Let $\operatorname{int} K_2 \neq \emptyset$. We first show (2) $\Longrightarrow$ (RCQ), so let (2) be satisfied. Then there exists $\varepsilon > 0$ such that

$$G(\bar{x}) + G'(\bar{x})h + \begin{pmatrix} 0 \\ B_{\varepsilon,Z_2}(0) \end{pmatrix} \subseteq K,$$

hence

$$\begin{pmatrix} 0 \\ B_{\varepsilon,Z_2}(0) \end{pmatrix} \subseteq G(\bar{x}) + G'(\bar{x})h - K \subseteq G(\bar{x}) + G'(\bar{x})X - K,$$

which implies (RCQ).

Next we show (RCQ) $\Longrightarrow$ (2). Assume that (2) does not hold, which means that the convex sets $G_2(\bar{x}) + G_2'(\bar{x})\big(G_1'(\bar{x})^{-1}[K_1 - G_1(\bar{x})]\big)$ and $\operatorname{int} K_2$ have empty intersection and we can separate them with a hyperplane $[z' = \alpha]$ to obtain

$$\langle z', G_2(\bar{x}) + G_2'(\bar{x})h - v\rangle_{Z^*,Z} \ge 0 \quad \text{for all } h \in G_1'(\bar{x})^{-1}[K_1 - G_1(\bar{x})],\ v \in K_2.$$

Choosing $z \in Z$ with $\langle z', z\rangle_{Z^*,Z} < 0$ and observing $\langle z', tz\rangle_{Z^*,Z} < 0$ for all $t > 0$ gives a contradiction to (RCQ) in the equivalent form (1), since we can choose t small enough such that $tv \in B_\varepsilon(0)$ for given ε, but from $\langle z', tz\rangle_{Z^*,Z} < 0$ we know that

$$tv \notin G_2(\bar{x}) + G_2'(\bar{x})\big(G_1'(\bar{x})^{-1}[K_1 - G_1(\bar{x})]\big) - K_2.$$

(c) (i) For $K_1 = \{0\}$ and the second set of constraints void, (RCQ) is just (recall $G_1(\bar{x}) \in K_1 = \{0\}$)

$$0 \in \operatorname{int}\big(G_1'(\bar{x})X\big). \tag{RCQ$_0$}$$

If $G_1'(\bar{x})$ is surjective, then $G_1'(\bar{x})X = Z_1$ and (RCQ$_0$) is trivially satisfied. Let conversely (RCQ$_0$) be true and let $z \in Z_1$. Choosing $\varepsilon > 0$ sufficiently small, there exists $h \in X$ such that

$$\varepsilon z = G_1'(\bar{x})h.$$

But then

$$z = G_1'(\bar{x})\varepsilon^{-1}h,$$

hence z lies in the range of $G_1'(\bar{x})$. Since $z \in Z_1$ was arbitrary, $G_1'(\bar{x})$ is surjective.

Since we now know that (RCQ) and thus the ACQ holds true, we have

$$T(\mathcal{F}, \bar{x}) = T_\ell(G_1, \{0\}, \bar{x}) = \Big\{d \in X \colon G_1'(\bar{x})d \in \operatorname{cone}\big(\{0\}, G_1(\bar{x})\big)\Big\}.$$

But $\operatorname{cone}(\{0\}, G_1(\bar{x})) = \{0\}$, hence the foregoing sets are exactly $\ker G_1'(\bar{x})$.

(ii) This is yet again exactly the same argument as in Lemma 3.17 in the lecture notes or in (ii). The representation for $T(\mathcal{F}, \bar{x})$ follows as in the foregoing exercise.

(iii) For a classical NLP, we identify as usual $X = \mathbb{R}^n$, $Z = Z_1 \times Z_2 = \mathbb{R}^p \times \mathbb{R}^m$, $G_1 = h \colon \mathbb{R}^n \to \mathbb{R}^p$ and $G_2 = g \colon \mathbb{R}^n \to \mathbb{R}^m$ as well as $K_1 = \{0\}^p$ as well as $K_2 = (-\infty, 0]^m$, where also clearly $\operatorname{int} K_2 = (-\infty, 0) \neq \emptyset$. Then the constraints are given exactly by $h(x) = 0$ and $g(x) \leq 0$. The foregoing exercise shows that (RCQ) for $\bar{x}$ satisfying $h(\bar{x}) = 0$ and $g(\bar{x}) \leq 0$ is equivalent $\nabla h(\bar{x})^T$ being surjective and the existence of $d \in \mathbb{R}^n$ such that

$$\nabla h(\bar{x})^T d = 0 \quad \text{and} \quad g(\bar{x}) + \nabla g(\bar{x})^T d < 0.$$

The latter translates to $g_i(\bar{x}) + \nabla g_i(\bar{x})^T d < 0$ for all $i = 1, \ldots, m$. For $g_i(\bar{x}) = 0$, i.e., for $i \in \mathcal{A}(\bar{x})$, this is true if and only if $\nabla g_i(\bar{x})^T d < 0$, which shows already the MFCQ. Conversely, assuming MFCQ, we can achieve $g_i(\bar{x}) + \nabla g_i(\bar{x})^T d < 0$ for $i \notin \mathcal{A}(\bar{x})$, where it could happen that $\nabla g_i(\bar{x})^T d > 0$, by scaling d appropriately as in Example 3.16 in the lecture notes.

Exercise 2 (Polar cone)**.** Let $\emptyset \neq C \subseteq X$ be a given set and consider its polar cone

$$C^\circ := \big\{x' \in X^* \colon \langle x', x\rangle_{X^*,X} \leq 0 \text{ for all } x \in C\big\}.$$

(a) Show that C° is a nonempty closed convex cone.

(b) Show that $\overline{C}^\circ = C^\circ$.

(c) Now assume that C is convex. Show that $C^\circ = \operatorname{cone}(C)^\circ$.

Solution.

(a) The polar cone C° is nonempty because clearly $0 \in C^\circ$. It is also closed as the (infinite) intersection of the closed convex sets

$$C^\circ = \bigcap_{x \in C} f_x^{-1}\big[(-\infty, 0]\big], \quad \text{where } f_x \in X^{**}\colon f_x(x') := \langle x', x\rangle.$$

Here, the sets $f_x^{-1}\big[(-\infty, 0]\big]$ are closed because they are the preimage of the closed set $(-\infty, 0]$ under the continuous function f_x, and they are convex because f_x is a linear mapping.

(b) Let $x' \in \overline{C}^\circ$ be given. Then $x' \in C^\circ$ follows from $C \subseteq \overline{C}$. Conversely, let $x' \in C^\circ$ and $x \in \overline{C}$. Then there exists a sequence $(x_k) \subseteq C$ such that $x_k \to x$ and

$$\big\langle x', x\big\rangle_{X^*,X} = \lim_{k\to\infty} \big\langle x', x_k\big\rangle_{X^*,X} \leq 0$$

due to continuity of x'. Hence, $x' \in \overline{C}^\circ$.

(c) Let $x' \in \operatorname{cone}(C)^\circ$. Then $x' \in C^\circ$ follows from $C \subseteq \operatorname{cone}(C)$. Conversely, let $x' \in C^\circ$ and $y \in \operatorname{cone}(C)$. Then there exist $\lambda > 0$ and $x \in C$ such that $y = \lambda x$ and thus

$$\big\langle x', y\big\rangle_{X^*,X} = \lambda \big\langle x', x\big\rangle_{X^*,X} \leq 0.$$

Exercise 3 (Interior of an important cone)**.** Let X be a function space over the set $\Omega \subset \mathbb{R}^n$ and consider the cone of nonpositive functions in X:

$$K_- := \Big\{ f \in X\colon f(x) \leq 0 \text{ for all } x \in \Omega \Big\}.$$

Determine whether K_- has nonempty interior for the choices $X = L^p(\Omega)$ for $1 \leq p \leq \infty$ and $X = C(\overline{\Omega})$.

Solution. Let $f \in K_-$.

1. Let $1 \leq p < \infty$ as well as $y \in \Omega$ and $\varepsilon > 0$ be given. We show that there exists a function g with $\|f - g\|_{L^p(\Omega)} \leq \varepsilon$, but $g \notin K_-$. Let ρ be small enough such that if $\|x - y\|_\infty < \rho$, then $x \in \Omega$, and set $\delta = \min(\rho, \varepsilon^{\frac{p}{n}})$ as well as

$$g\colon \Omega \to \mathbb{R}, \quad g(x) := f(x) + \chi_{B_{\delta,\infty}(y)}(x) = \begin{cases} f(x) + 1 & \text{if } \|x - y\|_\infty < \delta, \\ f(x) & \text{otherwise.} \end{cases}$$

Then we have $g \in L^p(\Omega)$ and

$$\|f - g\|^p_{L^p(\Omega)} = \int_{B_{\delta,\infty}(y)} 1 \,\mathrm{d}x = \delta^n \leq \varepsilon^p$$

due to the choice of $\delta \leq \varepsilon^{\frac{p}{n}}$. This shows that K_- has empty interior in $L^p(\Omega)$ for $1 \leq p < \infty$.

2. Let $\varepsilon > 0$ be arbitrary and assume that $f(x) < -\varepsilon$ for all (if $X = C(\overline{\Omega})$) or almost all (if $X = L^\infty(\Omega)$) $x \in \Omega$, respectively. Let $g \in B_{\varepsilon,X}(f)$. Then it follows that

$$g(x) = g(x) - f(x) + f(x) \le |g(x) - f(x)| + f(x) < \varepsilon - \varepsilon = 0 \quad \text{for (almost) all } x \in \Omega,$$

hence $g \in K_-$. This means that K_- has nonempty interior in the spaces $L^\infty(\Omega)$ or $C(\overline{\Omega})$.

Exercise 4 (Topological properties of convex sets)**.** Let $\emptyset \ne C \subseteq X$ be a convex set. For two points $x, y \in X$ we set

$$[x, y) = \{(1-\lambda)x + \lambda y \colon \lambda \in [0, 1)\},$$

so the set of convex combinations between x and y not including y.

(a) Show that $[x, y) \subset \operatorname{int} C$ for $x \in \operatorname{int} C$ and $y \in \overline{C}$. Infer that $\operatorname{int} C$ is convex.

Hint: It will be easier to first show the assertion for $y \in C$ and then extend the proof to $y \in \overline{C}$.

(b) Show that $\operatorname{int} C = \operatorname{int} \overline{C}$ if $\operatorname{int} C$ is nonempty.

Hint: There are multiple ways to solve this. Find at least two proofs. One possibility: Assume the contrary and use the geometric version of the Hahn-Banach theorem to construct a point $z \in \partial C$ with an open neighborhood whose intersection with C is empty (which is a contradiction, why?).

Remark: If the set C is even *convex-series* closed (*cs*-closed)—that means: For any sequences $(x_k) \subseteq C$ and $(\lambda_k) \ge 0$ with $\sum_{k=1}^\infty \lambda_k = 1$ for which $x = \sum_{k=1}^\infty \lambda_k x_k$ exists in X, we have $x \in C$—, then in fact $\operatorname{int} C = \operatorname{int} \overline{C}$. Such *cs*-closed sets are always trivially convex, and open or closed convex sets are also *cs*-closed. How does this fit with the assertion in (b)?

Solution.

(a) If $\operatorname{int} C = \emptyset$, there is nothing to prove. Otherwise, let $x \in \operatorname{int} C$ and $y \in \overline{C}$ and set $z := (1-\lambda)x + \lambda y$ for some $0 < \lambda < 1$. We need to show that there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \subseteq C$. So let $\varepsilon > 0$ be fixed for now, to be determined later, and choose some $v \in B_\varepsilon(z)$. In order so show that $v \in C$, we try to express it as a convex combination of elements of C.

Firstly assuming $y \in C$, we use y itself and some element $\bar{x}$ from a neighborhood of x. Let δ_x be such that $B_{\delta_x}(x) \subset X$ (this must exist because $x \in \operatorname{int} C$). We make the ansatz

$$v = (1-\alpha)\bar{x} + \alpha y \qquad \Longleftrightarrow \qquad \bar{x} = \frac{1}{1-\alpha}(v - \alpha y)$$

for some $0 < \alpha < 1$, also to be determined. In order to have $\bar{x} \in B_{\delta_x}(x)$, we calculate

$$\|x - \bar{x}\| = \frac{1}{1-\alpha}\|(1-\alpha)x - v + \alpha y\|$$

and we have $(1-\alpha)\bar{x}+\alpha y = z$ exactly if $\alpha := \lambda$. So of course we set this and obtain

$$\|x-\bar{x}\| = \frac{1}{1-\lambda}\|z-v\| < \frac{\varepsilon}{1-\lambda}.$$

This shows that with $\varepsilon := (1-\lambda)\delta_x$, we have $\bar{x} \in C$ and thus $v = (1-\lambda)\bar{x}+\lambda y \in C$. Since $v \in B_\varepsilon(z)$ was arbitrary, this shows $B_\varepsilon(z) \subset C$ and thus $z \in \operatorname{int} C$.

Now assume that only $y \in \overline{C}$. Since then possibly $y \notin C$, we cannot use y itself as an element to express z as a convex combination. However, if $y \notin C$, then $y \in \partial C$ and thus $B_{\delta_y}(y) \cap C \neq \emptyset$ for every $\delta_y > 0$ by the definition of a boundary point. We thus choose $\bar{y} \in B_{\delta_y}(y) \cap C$ and $\bar{x} \in B_{\delta_x}(x)$ with $\delta_x > 0$ as above and set, analogously to the first case above,

$$v = (1-\lambda)\bar{x}+\lambda\bar{y} \quad\iff\quad \bar{x} = \frac{1}{1-\lambda}(v-\lambda\bar{y}).$$

We have already chosen the λ-convex combination as above. Now calculate again

$$\|x-\bar{x}\| = \frac{1}{1-\lambda}\|(1-\lambda)x - v + \lambda\bar{y}\| = \frac{1}{1-\lambda}\|\underbrace{(1-\lambda)x+\lambda y}_{=z} - v + \lambda(\bar{y}-y)\|$$
$$\leq \frac{1}{1-\lambda}\Big(\|z-v\| + \lambda\|\bar{y}-y\|\Big).$$

Recall that $\|\bar{y}-y\| < \delta_y$ and that we are free to choose $\delta_y > 0$ because of the boundary point property for y. In particular, we are allowed to choose it depending on $\|z-v\|$, which we do by setting

$$\delta_y := \frac{1}{\lambda}\Big((1-\lambda)\frac{\delta_x}{2} - \|z-v\|\Big),$$

which makes $\|x-\bar{x}\| < \delta_x$. In order to have $\delta_y > 0$, this requires ε to be chosen as e.g. $\varepsilon := (1-\lambda)\frac{\delta_x}{2}$. Then we have altogether $v = (1-\lambda)\bar{x}+\lambda\bar{y}$ with $\bar{x},\bar{y} \in C$ and thus $v \in C$. Since $v \in B_\varepsilon(z)$ was arbitrary, we have $B_\varepsilon(z) \subset C$ and thus $z \in \operatorname{int} C$.

(b) It is clear that $\operatorname{int} C \subseteq \operatorname{int} \overline{C}$, so we only have to prove the reverse inclusion.

a) First proof following the hint: Assume that there is a point $x \in \operatorname{int} \overline{C} \setminus \operatorname{int} C$. Since by the first part of the exercise $\operatorname{int} C$ is convex and obviously open as well as nonempty by assumption, there exists a hyperplane $[f = \alpha]$ separating $\operatorname{int} C$ and x, i.e.,

$$\langle f, x\rangle_{X^*,X} \geq \alpha \geq \langle f, y\rangle_{X^*,X} \quad \text{for all } y \in C.$$

(The inequality is in fact true for all $y \in C$ instead of only $y \in \operatorname{int} C$ due to the non-strict separation.) Since x was assumed to be from $\operatorname{int} \overline{C}$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \overline{C}$. We set $D := B_\varepsilon(x) \cap [f > \alpha]$.

Then D is nonempty: Assume that $\langle f, \bar{y} \rangle \leq \alpha$ for all $\bar{y} \in B_\varepsilon(x)$, or equivalently $\langle f, x+y \rangle \leq \alpha = \langle f, x \rangle$ for all $y \in B_\varepsilon(0)$. Then $\langle f, y \rangle \leq 0$, which means that $\langle f, -y \rangle \geq 0$ and thus $\langle f, x-y \rangle \geq \alpha$ for all $y \in B_\varepsilon(0)$. From this it follows that $\langle f, x+y \rangle = \alpha$ and thus $\langle f, y \rangle = 0$ for all $y \in B_\varepsilon(0)$. This finally implies $f = 0$ in X^*, which is a contradiction to the definition of a hyperplane. Hence, D is nonempty.

Choose a point $z \in D$. Since D, as the intersection of two open sets, is open, there exists $\delta > 0$ such that $B_\delta(z) \subseteq D$. Moreover, $D \cap C = \emptyset$ by definition of D. On the other hand, $z \in B_\varepsilon(x) \subseteq \overline{C}$, hence $z \in \overline{C} \setminus C \subseteq \partial C$. But then $B_\delta(z)$ is an open neighborhood of the boundary point z of C which has empty intersection with C. Such a neighborhood cannot exist by the definition of a boundary point.

b) A more direct proof, using (a): Let $x \in \operatorname{int} \overline{C}$. Since $\operatorname{int} C \neq \emptyset$, there exists $y \in \operatorname{int} C$, and since $x \in \operatorname{int} \overline{C}$, there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \overline{C}$. We express x as a convex combination of $\bar{x} \in B_\varepsilon(x)$ and y with the ansatz

$$x = (1-\lambda)\bar{x} + \lambda y, \quad \text{for some } 0 < \lambda < 1,$$

so

$$\bar{x} = \frac{1}{1-\lambda}(x - \lambda y).$$

To calculate λ, we consider

$$\|x - \bar{x}\| = \frac{\lambda}{1-\lambda}\|y - x\| \overset{!}{<} \varepsilon,$$

so for instance

$$\lambda := \frac{1}{2}\frac{\varepsilon}{\|y-x\| + \varepsilon}$$

does the job. But then x is a convex combination of $\bar{x} \in \overline{C}$ and $y \in \operatorname{int} C$ and thus, by (a), $x \in \operatorname{int} C$.